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Generalizations of the Zero-Divisor Graph of a Ring

Shane Patrick Redmond
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Shane Patrick Redmond entitled "Generalizations of the Zero-Divisor Graph of a Ring." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

D. F. Anderson, Major Professor

We have read this dissertation and recommend its acceptance:

S. B. Mulay, David E. Dobbs, Michael W. Berry

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Interim Vice Provost
and Dean of the Graduate School

Generalizations of the Zero-Divisor Graph of a Ring

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Shane P. Redmond
May 2001

Dedication

To my past teachers.

Family, faculty, or

friend, I thank you all.

Acknowledgements

I would like to express my most sincere gratitude to my advisor, David F. Anderson. His patience, insight, understanding, and friendship through my graduate career have been appreciated.

I would also like to thank my Dissertation Committee, Michael W. Berry, David E. Dobbs, and Shashikant B. Mulay. In particular, I would like to thank Dr. Dobbs and Dr. Mulay who have taught me a great deal about algebra in their courses over the past years.

I would like to thank my family for their steadfast support as I pursued my doctoral degree. Finally, I would like to thank all my friends and fellow graduate students who, along with my family and advisors, have given me the encouragement I needed to reach my goals.

Abstract

Let R be a commutative ring with 1, and let $Z(R)$ denote the set of zero-divisors of R. We define an undirected graph $\Gamma(R)$ with vertices $Z(R)^* = Z(R) - \{0\}$, where distinct vertices x and y of R are connected if and only if $xy = 0$. This graph is called the zero-divisor graph of R. We extend the definition of the zero-divisor graph to a noncommutative ring in several ways. Next, given a commutative ring R and ideal I of R, we introduce the notion of an ideal-based graph. This is an undirected graph with vertex set $\{x \in R - I |\ xy \in I\ for\ some\ y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. The properties of such a graph are investigated. We give several results concerning the zero-divisor graph of a commutative ring. Finally, the appendix gives examples illustrating an equivalence relation on the vertices of $\Gamma(R)$ that can be used to produce a related graph for rings R of specific types.

Contents

List of Figures

Introduction

Let R be a commutative ring with 1. We let $Z(R)$ denote the set of zero-divisors of R. We define an undirected graph $\Gamma(R)$ with vertices $Z(R)^* = Z(R) - \{0\}$, where distinct vertices x and y of R are adjacent if and only if $xy = 0$.

The concept of a zero-divisor graph of a commutative ring was introduced by I. Beck in [**5**], which was mainly concerned with colorings of rings. The definition given above differs from that in earlier work of D. D. Anderson and M. Naseer [**1**] and Beck in that 0 is not taken as a vertex of $\Gamma(R)$. Several other works, including [**2**], [**3**], [**8**], [**13**], [**14**], [**15**], and [**16**], investigate the properties of the zero-divisor graph of a ring.

D. F. Anderson and P. S. Livingston [**2**] gave several fundamental results concerning $\Gamma(R)$ for a commutative ring R using the above definition. Chief among these is that, for a commutative ring R, $\Gamma(R)$ is *connected*, that is, there is a path of finite length along the edges of $\Gamma(R)$ between any two given distinct vertices. Also, as the next examples show, non-isomorphic rings may have isomorphic zero-divisor graphs. Figures 1 and 2 appear in [**2**]. Figure 3 is one of several graphs published for the first time in this context. (We often leave the labeling of the vertices to the reader.)

FIGURE 1. $\Gamma(\mathbb{Z}_9)$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$, or $\Gamma(\mathbb{Z}_3[X]/(X^2))$.

FIGURE 2. $\Gamma(\mathbb{Z}_6)$, $\Gamma(\mathbb{Z}_8)$, or $\Gamma(\mathbb{Z}_2[X]/(X^3))$.

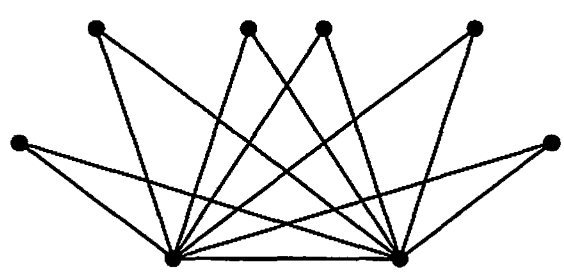

FIGURE 3. $\Gamma(\mathbb{Z}_{27})$.

We give here several notational conventions to be used throughout this work. Given a ring R, let $U(R)$ denote the units of R and, for $A \subseteq R$, let $A^* = A - \{0\}$. For any set X, we let $|X|$ be the number of elements of X if X is finite, and we write $|X| = \infty$ if X is infinite. Then we use the arithmetic conventions that $\infty + \infty = \infty$ and $n \cdot \infty = \infty$ for every $n > 0$. We denote the cardinality of a set X by $card(X)$. If Y is a subset of X, we let $X - Y$ denote the set-theoretic difference of sets. Definitions of previously introduced terms that are restated here are given in italics. Definitions appearing for the first time in this work are underlined when first defined. All rings considered are nonzero and contain an identity element $1 \neq 0$ unless otherwise noted. We often write $\mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}_n$.

Several definitions from graph theory are used throughout this work. Given a graph G, a *subgraph* H of G, denoted $H \subseteq G$, is a graph whose vertex set and edge set are subsets of those of G. A subgraph H of G is called an *induced subgraph* if all edges of G joining two vertices in H are also edges of H.

All paths between distinct vertices x and y of G are along a finite number of edges of G. For a graph G, let $d(x, y)$ be the length of the shortest path from x to a distinct y in G (and let $d(x, y) = \infty$ if no such path exists). The *diameter* of G is zero if G is the graph on one vertex and is $diam(G) = sup\{d(x, y) |\ x \text{ and } y \text{ are distinct vertices of } G\}$ otherwise.

A *cycle* in a graph G is a path that begins and ends at the same vertex. The *girth* of G, written $gr(G)$, is the length of the shortest cycle in G (and $gr(G) = \infty$ if G has no cycles).

The *degree* of a vertex x of G, written $deg(x)$, is the number of vertices of G adjacent to x. We never consider a vertex x of G as adjacent to itself.

A *complete graph* is a graph where all vertices are adjacent. The complete graph on n vertices is denoted by K^n. If G is a graph such that the vertices of G can be partitioned into two nonempty disjoint sets V_1 and V_2 such that vertices x and y are adjacent if and only if $x \in V_1$ and $y \in V_2$, then G is called a *complete bipartite graph.* A complete bipartite graph with disjoint vertex sets of size m and n, respectively, is denoted by $K^{m,n}$. We write $K^{n,\infty}$ (respectively, $K^{\infty,\infty}$) if one (respectively, both) of the disjoint vertex sets is infinite. A complete bipartite graph of the form $K^{1,n}$ is called a *star graph.* For a graph G, a complete subgraph is called a *clique.* The *clique number*, $\omega(G)$, is the greatest integer $n \geq 1$ such that $K^n \subseteq G$ (and $\omega(G) = \infty$ if G contains a subgraph isomorphic to K^n for each $n \geq 1$).

If G is a connected graph, the *connectivity* of G, denoted $\kappa(G)$, is the minimum number of vertices that it is necessary to remove from G in order to produce a disconnected graph. For a connected graph G, an edge E of G is a *bridge* if $G - E$ is disconnected. A vertex x of a connected graph G is a $cut-point$ if $G - \{x\}$ is not connected.

In Chapter 1, we develop several definitions for the zero-divisor graph of a noncommutative ring. The first of these gives us a directed graph. Conditions affecting whether this graph is connected or not connected include finiteness of the ring, whether the ring is artinian, and the existence of a two-sided identity. The other definitions in this chapter yield undirected graphs, which may or may not be connected. This work is the first effort to use noncommutative rings to generate zero-divisor graphs.

In Chapter 2, we generalize the notion of a zero-divisor graph of a ring R to that of an ideal-based graph. Given an ideal I of a commutative ring R, let $\Gamma_I(R)$ be the undirected graph with vertices $\{x \in R-I|\ xy \in I\ for\ some\ y \in R-I\}$, and let distinct vertices x and y be adjacent if and only if $xy \in I$. The construction of these graphs using "columns" of differing types and their relationship to $\Gamma(R/I)$ is examined. Several properties of these graphs are discussed, including girth, clique number, connectivity, bridges, and the degree of a vertex. A partial list of all such graphs on n vertices is given for certain values of n. An ordering on the vertices of $\Gamma_I(R)$ is introduced. This work is the first to define and study the ideal-based graph.

Chapter 3 gives several new results concerning the zero-divisor graph of a commutative ring. The definition of a zero-divisor graph is generalized to modules and rings without identity. The role of nilpotent elements is examined. The structure of direct products of rings and equivalence relations on the set of vertices of these graphs are discussed. All graphs generated by equivalence relations on the vertices of $\Gamma(R)$ appear here for the first time. Several results are given concerning the degree of a vertex.

CHAPTER 1

The Zero-Divisor Graph of a Noncommutative Ring

There are many ways to generalize the notion of the zero-divisor graph of a commutative ring to a noncommutative ring. In this chapter we investigate several of these.

Basic Notation and Definitions

DEFINITION 1. *For a ring R, we define the following subsets:*

(a.) Let $Z(R)$ denote the set of zero-divisors of R.

(b.) Let $Z_L(R)$ denote the set of left zero-divisors of R. That is,

$$Z_L(R) = \{x \in R|\ xa = 0 \ for\ some\ a \in R^*\}.$$

(c.) Let $Z_R(R)$ denote the set of right zero-divisors of R. That is,

$$Z_R(R) = \{x \in R|\ bx = 0 \ for\ some\ b \in R^*\}.$$

(d.) Let $Z_T(R)$ denote the set of two-sided zero-divisors of R. That is,

$$Z_T(R) = \{x \in R|\ xa = bx = 0 \ for\ some\ a, b \in R^*\}.$$

Note that $Z(R) = Z_L(R) \cup Z_R(R)$ and $Z_T(R) = Z_L(R) \cap Z_R(R)$. We thus have the following containment of sets.

$$Z(R) = Z_L(R) \cup Z_R(R) \supseteq \substack{Z_L(R) \\ Z_R(R)} \supseteq Z_L(R) \cap Z_R(R) = Z_T(R).$$

If R is a commutative ring, then all these sets are equal.

DEFINITION 2. *Let R be a ring. We define a (directed) graph* $\Gamma(R)$ *with vertices* $Z(R)^*$*, where* $x \to y$ *is an edge between distinct vertices* x *and* y *if and only if* $xy = 0$.

Then $\Gamma(R)$ is a directed graph on $Z(R)^*$. If R is a commutative ring, $y \to x$ is an edge whenever $x \to y$ is an edge. Therefore, if we view $\Gamma(R)$ as an undirected graph, this definition agrees with the usual definition of the zero-divisor graph of a commutative ring.

Figures 4, 5, 6, and 7 give examples of $\Gamma(R)$ for a noncommutative ring R. We leave the labeling of Figures 6 and 7 to the reader.

Noncommutative Rings With Identity

Throughout this section, all rings considered have a two-sided multiplicative identity element.

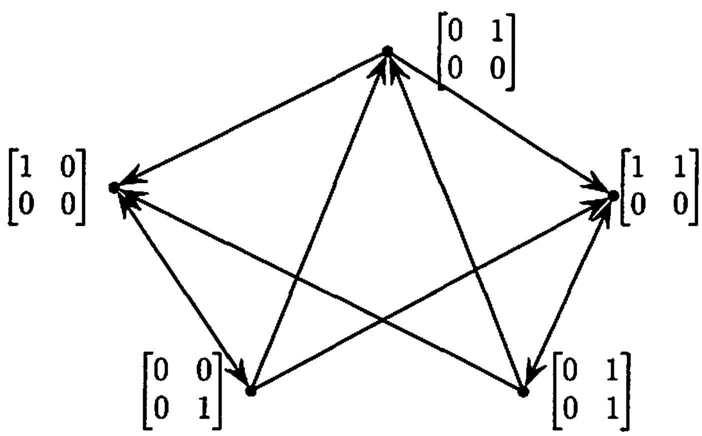

FIGURE 4. $\Gamma(R)$, where R is the ring of upper triangular 2×2 matrices over $\mathbb{Z}/2\mathbb{Z}$.

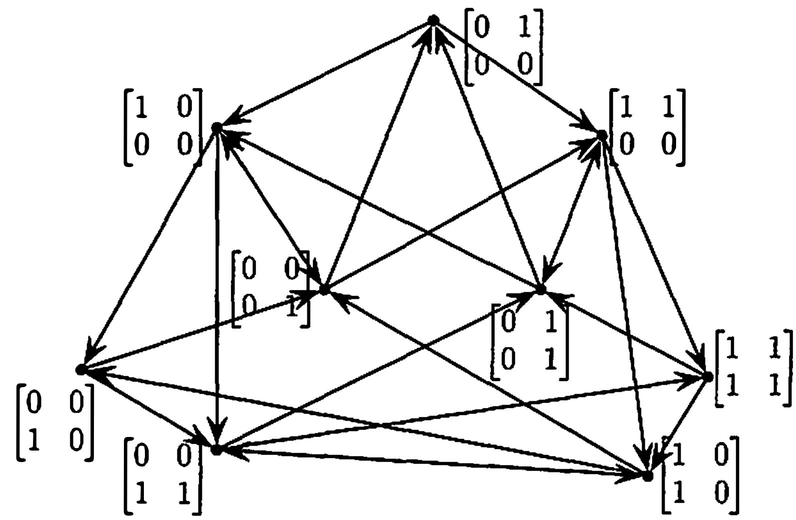

FIGURE 5. $\Gamma(R)$, where $R = M_2(\mathbb{Z}/2\mathbb{Z})$.

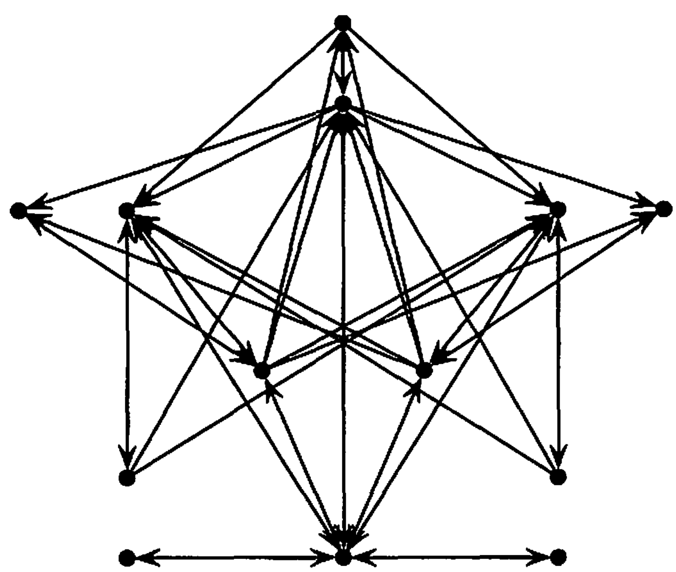

FIGURE 6. $\Gamma(R \times \mathbb{Z}/2\mathbb{Z})$, where R is the ring of upper triangular 2×2 matrices over $\mathbb{Z}/2\mathbb{Z}$.

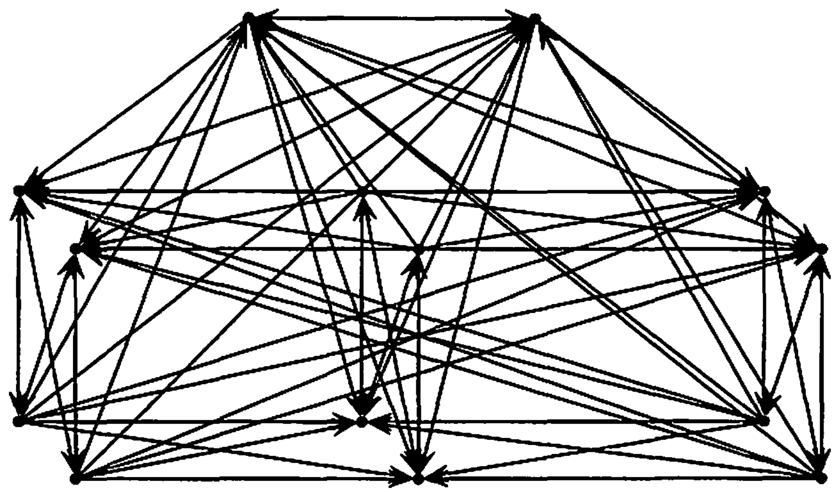

FIGURE 7. $\Gamma(R)$, where R is the ring of upper triangular 2×2 matrices over $\mathbb{Z}/3\mathbb{Z}$.

We say that a directed graph G is *connected* if there is a path following the directed edges of G from any vertex x of G to any other vertex y of G. Unlike the case for a commutative ring, $\Gamma(R)$ need not be connected if R is noncommutative, as seen in the next example.

EXAMPLE 3. *Let K be a field, and let $V = \oplus_{i=1}^{\infty} K$. Let $R = Hom_K(V, V)$. Under point-wise addition and multiplication taken to be composition of functions, R is an infinite noncommutative ring with identity. Let $\pi_1 : V \to V$ be defined by $(a_1, a_2, \ldots) \mapsto (a_1, 0, 0, \ldots)$ and $f : V \to V$ be defined by $(a_1, a_2, \ldots) \mapsto (0, a_1, a_2, \ldots)$. Then $\pi_1, f \in R$ and $\pi_1 f = 0$, while $f \pi_1 \neq 0$. Clearly f is not a left zero-divisor in R (if $g \in R$ with $fg(a_1, a_2, \ldots) = (0, 0, \ldots)$ for all $(a_1, a_2, \ldots) \in V$, then $g(a_1, a_2, \ldots) = (0, 0, \ldots)$ for all $(a_1, a_2, \ldots) \in V$). Hence, $\Gamma(R) \neq \emptyset$ and $\Gamma(R)$ is not connected since there is no path leading from the vertex f to any other vertex of $\Gamma(R)$.*

REMARK 4. *Note that the structure of* $\Gamma(R)$ *in Example 3 is still rather rich. By defining* π_j *to be* $(a_1, a_2, \dots) \mapsto (0, \dots, 0, a_j, 0, \dots)$*, the vertices* $\{\pi_j\}_{j=1}^{\infty}$ *comprise a complete subgraph of* $\Gamma(R)$.

THEOREM 5. *Let* R *be a ring. Then* $Z_L(R) = Z_R(R)$ *if and only if* $\Gamma(R)$ *is connected. Moreover, if* $\Gamma(R)$ *is connected, then* $diam(\Gamma(R)) \leq 3$.

Proof : Suppose that $Z_L(R) = Z_R(R)$.
Let x and y be distinct vertices of $\Gamma(R)$. (Then $x \neq 0$ and $y \neq 0$.)
Case 1: $xy = 0$. Then $x \to y$ is a path.
Case 2: $xy \neq 0$ and $x^2 = 0$ and $y^2 = 0$. Then $x \to xy \to y$ is a path.
Case 3: $xy \neq 0$, $y^2 \neq 0$, and $x^2 = 0$. Then there exists $b \in R - \{x, y, 0\}$ such that $by = 0$. If $xb = 0$, then $x \to b \to y$ is a path. If $xb \neq 0$, then $x \to xb \to y$ is a path.
Case 4: $xy \neq 0, x^2 \neq 0$, and $y^2 = 0$. Then there exists $a \in R - \{x, y, 0\}$ such that $xa = 0$. If $ay = 0$, then $x \to a \to y$ is a path. If $ay \neq 0$, then $x \to ay \to y$ is a path.
Case 5: $xy \neq 0$, $y^2 \neq 0$, and $x^2 \neq 0$. There exist $a \in R - \{x, y, 0\}$ such that $xa = 0$ and $b \in R - \{x, y, 0\}$ such that $by = 0$.
Subcase 1: $a = b$. Then $x \to a \to y$ is a path.
Subcase 2: $a \neq b$. If $ab = 0$, then $x \to a \to b \to y$ is a path. If $ab \neq 0$, then $x \to ab \to y$ is a path.
Thus $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$.

Conversely, suppose that $\Gamma(R)$ is connected. Let $0 \neq x \in Z_R(R)$. Then x is a vertex of $\Gamma(R)$. If x is the only vertex of $\Gamma(R)$, then $x^2 = 0$ and $Z_R(R) = Z_L(R) = \{x, 0\}$. So let y be another vertex of $\Gamma(R)$.

Since the graph is connected, there is a path $x \to a_1 \to \cdots \to a_n \to y$ in $\Gamma(R)$. Therefore, $xa_1 = 0$ and thus $x \in Z_L(R)$. If $0 \neq x \in Z_L(R)$, then there is a path $y \to b_1 \to \cdots \to b_m \to x$ in $\Gamma(R)$. Therefore, $b_m x = 0$ and so $x \in Z_R(R)$. Thus $Z_L(R) = Z_R(R)$. □

COROLLARY 6. *Let R be a left- and right-artinian ring with a two-sided identity. Then* $Z_L(R) = Z_R(R)$. *Moreover,* $\Gamma(R)$ *is connected with* $diam(\Gamma(R)) \leq 3$.

Proof : Let $a \in R - Z_R(R)$. Then the map $r \mapsto ra$ is an injective homomorphism of R into R. Since R is artinian, this map is also surjective by Lemma 11.6 of [4]. Thus, there is some $s \in R$ such that $sa = 1$. Therefore, $a \notin Z_L(R)$ (for if $at = 0$, then $t = 1t = (sa)t = s(at) = 0$).

By a similar argument, if $b \in R - Z_L(R)$, then $b \notin Z_R(R)$. Hence, $Z_L(R) = Z_R(R)$. The last statement follows from Theorem 5. □

REMARK 7. *A special case of the last corollary arises if R is a finite ring with two-sided identity. Then* $\Gamma(R)$ *is finite and connected. N. Ganesan has shown that if a ring has a finite number of zero-divisors, then the ring is finite* [10]. *Hence, for any ring R with two-sided identity,* $\Gamma(R)$ *is finite also implies* $\Gamma(R)$ *is connected.*

EXAMPLE 8. *We may have* $Z_R(R) = Z_L(R)$ *for an infinite noncommutative ring R as well. Let* $\mathbb{H}$ *be the ring of quaternions over the field of real numbers. Then* $\mathbb{H}$ *is an infinite noncommutative division ring. Let* $R = \mathbb{H} \times \mathbb{H}$. *The only nonzero zero-divisors of R are of the form (0,t) or (t,0) for* $0 \neq t \in \mathbb{H}$. *Note then that all zero-divisors of R are two-sided(i.e.,* $Z_L(R) = Z_R(R)$*).*

DEFINITION 9. *Let R be a ring. We define a graph $\overline{\Gamma(R)}$ with vertices $Z(R)^*$, where distinct vertices x and y are adjacent if and only if either $xy = 0$ or $yx = 0$.*

Then $\overline{\Gamma(R)}$ is an undirected graph on $Z(R)^*$. Indeed, the only difference between $\Gamma(R)$ and $\overline{\Gamma(R)}$ is the fact that the former is a directed graph and the latter is undirected (that is, the graphs share the same vertices and the same edges if directions on the edges are ignored). If R is a commutative ring, this definition agrees with the previous definition of the zero-divisor graph.

THEOREM 10. *Let R be a ring. Then $\overline{\Gamma(R)}$ is a connected graph and $diam(\overline{\Gamma(R)}) \leq 3$.*

Proof : Let x and y be distinct vertices of $\overline{\Gamma(R)}$.

<u>Case 1:</u> $xy = 0$ or $yx = 0$. Then $x - y$ is a path.

Suppose $xy \neq 0$ and $yx \neq 0$.

<u>Case 2:</u> $x^2 = y^2 = 0$. Then $x - xy - y$ is a path.

<u>Case 3:</u> $x^2 = 0$ and $y^2 \neq 0$. Then there is some $b \in R - \{x, y, 0\}$ such that either $by = 0$ or $yb = 0$. If either $xb = 0$ or $bx = 0$, then $x - b - y$ is a path. If $xb \neq 0$ and $bx \neq 0$, then $x - bx - y$ is a path if $yb = 0$ and $x - xb - y$ is a path if $by = 0$.

<u>Case 4:</u> $x^2 \neq 0$ and $y^2 = 0$. We can use an argument similar to that of the above case to obtain a path.

<u>Case 5:</u> $x^2 \neq 0$ and $y^2 \neq 0$. Then there exist $a, b \in R - \{0, x, y\}$ such that either $ax = 0$ or $xa = 0$ and such that either $by = 0$ or $yb = 0$. If

$a = b$, then $x - a - y$ is a path. If $ab = 0$ or $ba = 0$, then $x - a - b - y$ is a path. So suppose $ab \neq 0$, $ba \neq 0$, and $a \neq b$.

Subcase 1: $x - a - y$ is a path if $ay = 0$ or $ya = 0$.

Subcase 2: $x - ab - y$ is a path if $xa = 0$ and $by = 0$.

Subcase 3: $x - ba - y$ is a path if $ax = 0$ and $yb = 0$.

Subcase 4: $x - ay - b - y$ is a path if $xa = 0$, $yb = 0$, and $ay \neq 0$.

Subcase 5: $x - ya - b - y$ is a path if $ax = 0$, $by = 0$, and $ya \neq 0$.

Thus $\overline{\Gamma(R)}$ is connected and $diam(\overline{\Gamma(R)}) \leq 3$. □

DEFINITION 11. *Let R be a ring. We define a graph $\Gamma'(R)$ with vertices $Z_T(R)^*$, where distinct vertices x and y are adjacent if and only if both $xy = 0$ and $yx = 0$.*

Then $\Gamma'(R)$ is an undirected graph on $Z_T(R)^*$.

DEFINITION 12. *Let R be a ring. We define a graph $\Gamma''(R)$ with vertices $Z(R)^*$, where distinct vertices x and y are adjacent if and only if both $xy = 0$ and $yx = 0$.*

Then $\Gamma''(R)$ is an undirected graph on $Z(R)^*$. If R is commutative, both $\Gamma'(R)$ and $\Gamma''(R)$ agree with the usual definition of the zero-divisor graph. Note that both $\Gamma'(R)$ and $\Gamma''(R)$ are subgraphs of $\overline{\Gamma(R)}$, but neither $\Gamma'(R)$ nor $\Gamma''(R)$ need be connected. Examples are given in Figure 8 and in the next section.

PROPOSITION 13. *Let R be a ring. Then $\Gamma'(R)$ is an induced subgraph of $\Gamma''(R)$.*

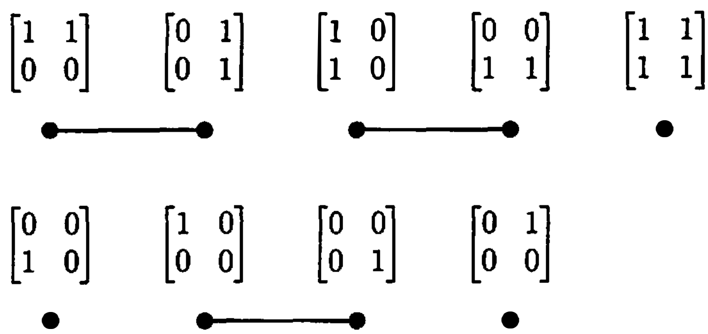

FIGURE 8. $\Gamma''(R)$, where $R = M_2(\mathbb{Z}/2\mathbb{Z})$. Since R is a finite ring with a two-sided identity, $\Gamma''(R) = \Gamma'(R)$.

Proof : Clearly the vertex set of $\Gamma'(R)$ is contained in that of $\Gamma''(R)$. Since the relationship defining adjacency in $\Gamma'(R)$ and $\Gamma''(R)$ is the same, the result follows. □

Noncommutative Rings With No Two-Sided Identity

If R is a noncommutative ring without a two-sided multiplicative identity, then the conclusion of Corollary 6 may fail. If R is a finite ring without a two-sided identity, we may have $Z_L(R) \neq Z_R(R)$, and therefore $\Gamma(R)$ would not be connected as a directed graph. However, even in this case, $\overline{\Gamma(R)}$ is always connected as an undirected graph. The proof of this fact is exactly the same as in Theorem 10. Note that in Figures 9 and 10, $\overline{\Gamma(R)}$ is connected even though there exist in each

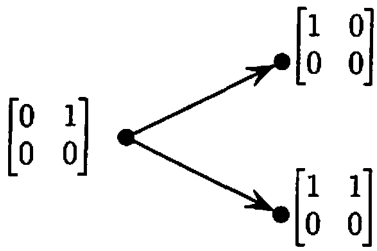

FIGURE 9. $\Gamma(R)$, where $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$.

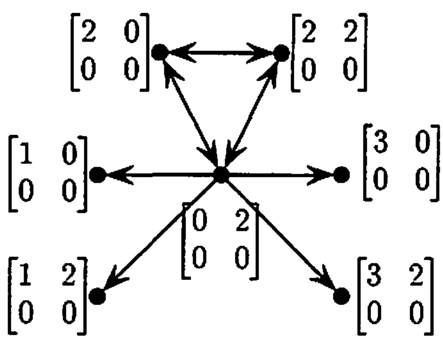

FIGURE 10. $\Gamma(R)$, where $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{Z}_4,\ b \in 2\mathbb{Z}_4 \right\}$.

case distinct vertices x and y with no path from x to y or from y to x in $\Gamma(R)$.

Note that in Figure 9, $Z_L(R) = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ and $Z_R(R) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$.

Figure 11 gives an example of a finite noncommutative ring R without identity such that $\Gamma(R)$ is connected. Figures 12 and 13 feature a ring R such that $\Gamma'(R) \neq \Gamma''(R)$.

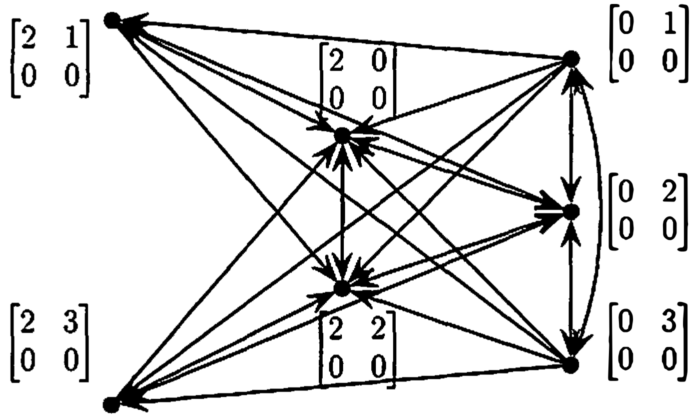

FIGURE 11. $\Gamma(R)$, where $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a \in 2\mathbb{Z}_4,\ b \in \mathbb{Z}_4 \right\}$.

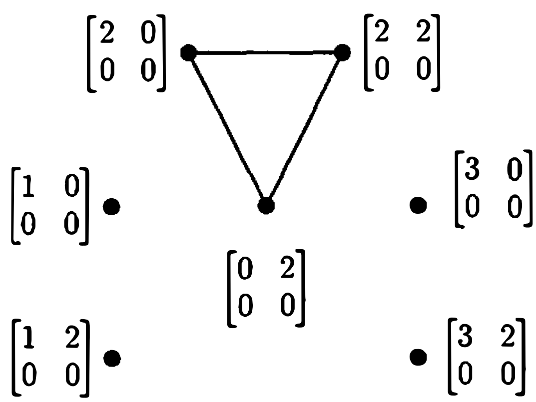

FIGURE 12. $\Gamma''(R)$, where $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{Z}_4,\ b \in 2\mathbb{Z}_4 \right\}$.

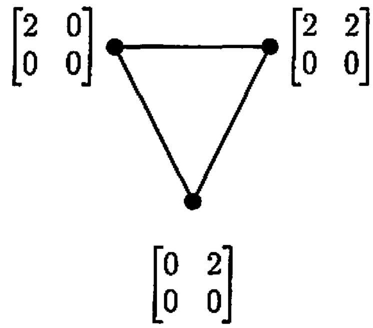

FIGURE 13. $\Gamma'(R)$, where $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{Z}_4,\ b \in 2\mathbb{Z}_4 \right\}$.

CHAPTER 2

An Ideal-Based Zero-Divisor Graph

In this chapter, we generalize the notion of a zero-divisor graph to a graph based on a nonzero ideal. All rings in this chapter are commutative with (nonzero) identity.

Definition and Basic Structure

DEFINITION 1. *Let R be a commutative ring and let I be an ideal of R. We define an undirected graph* $\Gamma_I(R)$ *with vertices* $\{x \in R - I \mid xy \in I \ for \ some \ y \in R - I\}$, *where distinct vertices* x *and* y *are adjacent if and only if* $xy \in I$.

PROPOSITION 2. *(a.) If* $I = (0)$, *then* $\Gamma_I(R) = \Gamma(R)$.
(b.) Let I be a proper ideal of R. Then $\Gamma_I(R) = \emptyset$ *if and only if I is a prime ideal of R.*

Proof : (a) This is clear.

(b) Suppose that I is a prime ideal of R. Then $xy \in I$ implies $x \in I$ or $y \in I$. Hence the vertex set of $\Gamma_I(R)$ is empty.

Conversely, suppose that $\Gamma_I(R) = \emptyset$. Therefore, if $x \in R - I$ and $xy \in I$ for some $y \in R$, we must have $y \in I$ (otherwise, x is a vertex of $\Gamma_I(R)$). Hence I is a prime ideal of R. □

Note that Proposition 2 (b) is equivalent to saying $\Gamma_I(R) = \emptyset$ if and only if R/I is a domain. That is, $\Gamma_I(R) = \emptyset$ if and only if $\Gamma(R/I) = \emptyset$.

We explore the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$ throughout this section.

To avoid trivialities, we always assume $I \neq R$ for an ideal I of R.

REMARK 3. *Note that in Figures 14 and 15, $R/I \simeq \mathbb{Z}_6$ and $S/J \simeq \mathbb{Z}_8$. Then $\Gamma(R/I) \simeq \Gamma(S/J)$, both being the graph on three vertices with two edges. Thus we have an example where $\Gamma(R/I) \simeq \Gamma(S/J)$, but $\Gamma_I(R) \not\simeq \Gamma_J(S)$.*

THEOREM 4. *Let I be an ideal of a ring R. Then $\Gamma_I(R)$ is connected with diam$(\Gamma_I(R)) \leq 3$. Furthermore, if $\Gamma_I(R)$ contains a cycle, then gr$(\Gamma_I(R)) \leq 7$.*

Proof : Let x and y be distinct vertices of $\Gamma_I(R)$.
Case 1: $xy \in I$. Then $x - y$ is a path in $\Gamma_I(R)$.
Case 2: $xy \notin I$, $x^2 \in I$, and $y^2 \in I$. Then $x - xy - y$ is a path.

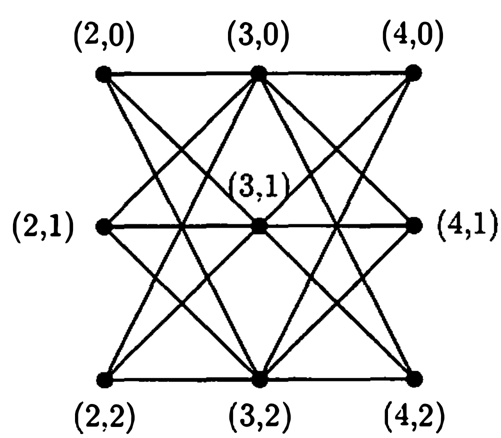

FIGURE 14. $\Gamma_I(R)$, where $R = \mathbb{Z}_6 \times \mathbb{Z}_3$ and $I = 0 \times \mathbb{Z}_3$.

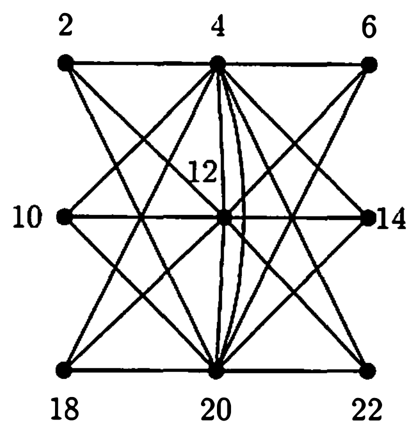

FIGURE 15. $\Gamma_J(S)$, where $S = \mathbb{Z}_{24}$ and $J = (8)$.

Case 3: $xy \notin I, x^2 \in I$, and $y^2 \notin I$. Then there is some $b \in R - I$ such that $by \in I$. If $bx \in I$, then $x - b - y$ is a path. If $bx \notin I$, then $x - bx - y$ is a path.

Case 4: $xy \notin I, y^2 \in I$, and $x^2 \notin I$. Then we obtain a path as in the above case.

Case 5: $xy \notin I, x^2 \notin I$, and $y^2 \notin I$. Then there exist $a, b \in R - (I \cup \{x, y\})$ such that $ax \in I$ and $by \in I$. If $a = b$, then $x - a - y$ is a path. If $a \neq b$ and $ab \in I$, then $x - a - b - y$ is a path. If $a \neq b$ and $ab \notin I$, then $x - ab - y$ is a path.

Thus $\Gamma_I(R)$ is connected and $diam(\Gamma_I(R)) \leq 3$. For any undirected graph G, $gr(G) \leq 2diam(G) + 1$ whenever G contains a cycle (see, for example, [6]). Thus $gr(\Gamma_I(R)) \leq 7$. $\square$

The next several results investigate the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$.

THEOREM 5. *Let I be an ideal of a ring R, and let $x, y \in R - I$. Then:*

(a.) If $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$, then x is adjacent to y in $\Gamma_I(R)$.

(b.) If x is adjacent to y in $\Gamma_I(R)$ and $x + I \neq y + I$, then $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$.

(c.) If x is adjacent to y in $\Gamma_I(R)$ and $x + I = y + I$, then $x^2, y^2 \in I$.

Proof : (a) $x + I$ adjacent to $y + I$ in $\Gamma(R/I)$ implies $xy + I = (x + I)(y + I) = 0 + I$. Thus $xy \in I$.

(b) Suppose that $x + I \neq y + I$. Then $xy \in I$ since x is adjacent to y. Therefore, $(x + I)(y + I) = xy + I = 0 + I$. Hence, $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$.

(c) Suppose that $x + I = y + I$. Then $(x^2 + I) = (x + I)(y + I) = xy + I = 0 + I$. Thus $x^2 \in I$, and similarly $y^2 \in I$. □

COROLLARY 6. *If x and y are (distinct) adjacent vertices in $\Gamma_I(R)$, then all (distinct) elements of $x + I$ and $y + I$ are adjacent in $\Gamma_I(R)$. If $x^2 \in I$, then all the distinct elements of $x + I$ are adjacent in $\Gamma_I(R)$.*

For a graph G, we say $\{G_\delta\}_{\delta \in \Delta}$ is a collection of *disjoint subgraphs* of G if all the vertices and edges of each G_δ are contained in G and no two of these G_δ contain a common vertex.

COROLLARY 7. *Let I be an ideal of a ring R. Then $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$.*

Proof : Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of coset representatives of the vertices of $\Gamma(R/I)$; that is, $\cup_{\lambda\in\Lambda}\{a_\lambda + I\} = Z(R/I)^*$, and if $\lambda \neq \beta$, then $a_\lambda + I \neq a_\beta + I$. For each $i \in I$, define a graph G_i with vertices $\{a_\lambda + i | \lambda \in \Lambda\}$, where $a_\lambda + i$ is adjacent to $a_\beta + i$ in G_i whenever $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(R/I)$; i.e., whenever $a_\lambda a_\beta \in I$. By the above theorem, G_i is a subgraph of $\Gamma_I(R)$. Also, each $G_i \simeq \Gamma(R/I)$, and G_i and G_j contain no common vertices if $i \neq j$. □

Clearly there is a strong relationship between $\Gamma(R/I)$ and $\Gamma_I(R)$. The next theorem tells us how we can explicitly construct $\Gamma_I(R)$ from $\Gamma(R/I)$.

THEOREM 8. *Let I be an ideal of a ring R. We construct a graph G as follows:*
Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of coset representatives of the vertices of $\Gamma(R/I)$. For each $i \in I$, define a graph G_i with vertices $\{a_\lambda + i | \lambda \in \Lambda\}$, where edges are defined by the relationship $a_\lambda + i$ is adjacent to $a_\beta + i$ in G_i if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(R/I)$ (i.e., $a_\lambda a_\beta \in I$). Define the graph G to have as its vertex set $V = \bigcup_{i\in I} G_i$. We define the edge set of G to be: (1.) all edges contained in G_i for each $i \in I$, (2.) for distinct $\lambda, \beta \in \Lambda$ and for any $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\beta + j$ if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(R/I)$ (i.e., $a_\lambda a_\beta \in I$), (3.) for $\lambda \in \Lambda$ and distinct $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\lambda + j$ if and only if $a_\lambda^2 \in I$.
Then $G = \Gamma_I(R)$.

Proof : Clearly V is contained in the vertex set of $\Gamma_I(R)$. Note that for any vertex x of $\Gamma_I(R)$, $x + I$ is a zero-divisor of R/I by Theorem 5.

Therefore, the vertices of $\Gamma_I(R)$ are contained in V. By Theorem 5, all edges of types 1 and 2 defined above are also edges of $\Gamma_I(R)$. If $a_\lambda + i$ is adjacent to $a_\lambda + j$ in G for distinct $i, j \in I$, then $a_\lambda^2 \in I$. Therefore, $(a_\lambda + i)(a_\lambda + j) = a_\lambda^2 + ia_\lambda + ja_\lambda + ij \in I$. Thus, the edges of type 3 defined above are also edges of $\Gamma_I(R)$.

Let x and y be distinct vertices of $\Gamma_I(R)$ with x adjacent to y. There exist $i, j \in I$ and $\lambda, \beta \in \Lambda$ such that $x = a_\lambda + i$ and $y = a_\beta + j$. If $\lambda \neq \beta$, x adjacent to y implies $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(R/I)$ by Theorem 5. Hence, the edge $x - y$ corresponds to an edge of type 1 or 2 of G. If $\lambda = \beta$, then $xy = (a_\lambda + i)(a_\lambda + j) = a_\lambda^2 + ia_\lambda + ja_\lambda + ij \in I$. Thus $a_\lambda^2 \in I$, and the edge $x - y$ corresponds to an edge of type 3 of G. $\square$

COROLLARY 9. *Let I be an ideal of a ring R. Then $\Gamma_I(R)$ is a graph on a finite number of vertices if and only if either R is finite or I is a prime ideal.*

Proof : If I is a prime ideal, then $\Gamma_I(R) = \emptyset$ by Proposition 2. Clearly, the vertex set of $\Gamma_I(R)$ is finite if R is finite.

Conversely, suppose that $\Gamma_I(R)$ is a graph on a finite number of vertices and I is not a prime ideal. Then also, $\Gamma(R/I)$ is a graph on a finite number of vertices and I is finite. By Theorem 2.2 of [**2**], $\Gamma(R/I)$ is finite implies that R/I is finite. Since R/I is finite and I is finite, R must also be finite. $\square$

LEMMA 10. *Let I be an ideal of a ring R. If $\Gamma(R/I)$ is infinite, then $\Gamma_I(R)$ is infinite. If $\Gamma(R/I)$ is a graph on N vertices, the $\Gamma_I(R)$ is a graph on $N \cdot |I|$ vertices.*

Proof : This is immediate from Theorem 8. $\square$

DEFINITION 11. *Using the notation as in the above theorem, we call the subset $a_\lambda + I$ a column of $\Gamma_I(R)$. If $a_\lambda^2 \in I$, then we call $a_\lambda + I$ a connected column of $\Gamma_I(R)$.*

Note that if R/I is reduced, then $\Gamma_I(R)$ has no connected columns. In a later section, we examine the types of columns an ideal-based graph of a ring may have.

The method of Theorem 8 can be used to construct $\Gamma_I(R)$. Figure 16 illustrates this using $\Gamma_I(\mathbb{Z}_{24})$ with $I = (8)$ (as in Figure 15). Note that $\mathbb{Z}_{24}/I \simeq \mathbb{Z}_8$. Also, the columns have height $|I| = 3$. We start by stacking $|I| = 3$ copies of $\Gamma(\mathbb{Z}_8)$, creating edge set 1 of the theorem. Then we connect adjacent columns to create edge set 2. Finally, we place edges in the connected column $4 + (8)$ to create edge set 3.

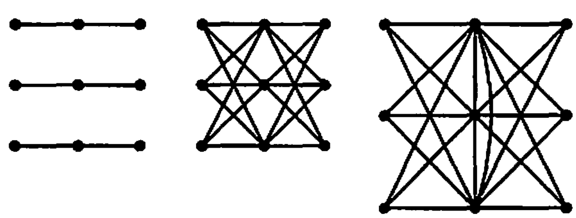

FIGURE 16. The stages used to construct Figure 15.

The construction of Figure 14 is similar, except that this graph has no connected columns.

PROPOSITION 12. *Let $R \subseteq S$ be rings and let I be an ideal of S. Then $I \cap R$ is an ideal of R and $\Gamma(R/(I \cap R))$ is an induced subgraph of $\Gamma(S/I)$.*

Proof : Clearly $I \cap R$ is an ideal of R and $R - (R \cap I) \subseteq S - I$. Let $x + I \cap R$ be adjacent to $y + I \cap R$ in $\Gamma(R/(I \cap R))$. Then $xy + I \cap R = (x + I \cap R)(y + I \cap R) = 0 + I \cap R$. So $xy \in I \cap R \subseteq I$. Thus $(x + I)(y + I) = 0 + I$, where $x + I$ is adjacent to $y + I$ in $\Gamma(S/I)$.

Conversely, if $y + I$ and $x + I$ are adjacent vertices of $\Gamma(S/I)$ with $x, y \in R$, then $xy \in I \cap R$. Therefore, $x + I \cap R$ and $y + I \cap R$ are adjacent in $\Gamma(R/(I \cap R)$. □

PROPOSITION 13. *Let $R \subseteq S$ be rings and let I be an ideal of S. Then $\Gamma_{I \cap R}(R)$ is an induced subgraph of $\Gamma_I(S)$.*

Proof : This follows from the fact that $I \cap R$ is an ideal of R and $R - (R \cap I) \subseteq S - I$. □

PROPOSITION 14. *Let I be an ideal of the ring R ($I \neq R$) and let J be an ideal of the ring S. If card$(J) =$ card$(I \times J)$, then $\Gamma_{I \times J}(R \times S) \simeq \Gamma_{0 \times J}(R/I \times S)$.*

Proof : First note that *card*$(I \times J) =$ *card*$(0 \times J)$, so the columns of the two graphs have the same height. By the first isomorphism theorem, $\frac{R \times S}{I \times J} \simeq \frac{R/I \times S}{0 \times J}$. Therefore, $\Gamma(\frac{R \times S}{I \times J}) \simeq \Gamma(\frac{R/I \times S}{0 \times J})$. It only remains to verify that the connected columns of $\Gamma_{I\times J}(R\times S)$ correspond to the connected columns of $\Gamma_{0\times J}(R/I\times S)$.

Suppose $\overline{x}=(a,b)+(I\times J)$ is a connected column of $\Gamma_{I\times J}(R\times S)$. Then $\overline{x}^2=\overline{0}$ in $\frac{R\times S}{I\times J}$. In particular, we have $a^2\in I$ and $b^2\in J$. Therefore, $(a+I)^2=0+I\in R/I$. Hence, $(a+I,b)+(0\times J)$ is a connected column in $\Gamma_{0\times J}(R/I\times S)$ since this element squares to zero in $\frac{R/I\times S}{0\times J}$.

Suppose $\overline{y}=(c+I,d)+(0\times J)$ is a connected column of $\Gamma_{0\times J}(R/I\times S)$. Then $\overline{y}^2=\overline{0}$ in $\frac{R/I\times S}{0\times J}$. Therefore, $c^2\in I$ and $d^2\in J$. Hence, $(c,d)+(I\times J)$ is a connected column in $\Gamma_{I\times J}(R\times S)$ since this element squares to zero in $\frac{R\times S}{I\times J}$. □

COROLLARY 15. *If $card(I)=card(J)=card(I\times J)$, then $\Gamma_{I\times 0}(R\times S/J)\simeq\Gamma_{0\times J}(R/I\times S)$.*

Connectivity

DEFINITION 16. *A vertex x of a connected graph G is a <u>cut − point</u> of G if there are vertices u, w of G such that x is in every path from u to w (and $x\neq u$, $x\neq w$). Equivalently, for a connected graph G, x is a cut-point of G if $G-\{x\}$ is not connected.*

THEOREM 17. *If I is a nonzero proper ideal of R, then $\Gamma_I(R)$ has no cut-points.*

Proof : Assume the vertex x of $\Gamma_I(R)$ is a cut-point. Then there exist vertices $u,w\in R-I$ such that x lies on every path from u to w. By Theorem 4, the shortest path from u to w is of length 2 or 3.

<u>*Case 1 :*</u> Suppose $u - x - w$ is a path of shortest length from u to w. If $x + I = u + I$, then x adjacent w implies u is adjacent to w by Corollary 6. Similarly, if $x + I = w + I$, u is adjacent to w. So suppose $x + I \neq u + I$ and $x + I \neq w + I$. Let $0 \neq i \in I$. Then $ux, xw \in I$ imply $u(x+i), w(x+i) \in I$. Hence $u - (x+i) - w$ is a path in $\Gamma_I(R)$. Thus in all cases we get a contradiction.

<u>*Case 2 :*</u> Suppose (without loss of generality) $u - x - y - w$ is a path of shortest length from u to w in $\Gamma_I(R)$. If $x + I = y + I$, then u adjacent to x implies u is adjacent to y and therefore $u - y - w$ is a path. If $x + I \neq y + I$, then let $0 \neq i \in I$. As above, u and y adjacent to x means that u and y are also adjacent to $x + i$. Hence $u - (x+i) - y - w$ is a path. Thus in all cases we get a contradiction. □

The *connectivity* $\kappa(G)$ of a graph G is defined to be the minimum number of vertices it is necessary to remove from G in order to produce a disconnected graph. We provide bounds on $\kappa(\Gamma_I(R))$ for a given ring R and ideal I of R. Recall that if $I = (0)$, then $\Gamma(R) = \Gamma_I(R)$.

THEOREM 18. *Let I be a nonzero proper ideal of a ring R.*
(a.) If $\Gamma(R/I)$ is the graph on one vertex, then $\kappa(\Gamma_I(R)) = |I| - 1$.
(b.) If $\Gamma(R/I)$ has at least two vertices, then
$1 \leq \kappa(\Gamma_I(R)) \leq |I| \cdot \kappa(\Gamma(R/I))$.
(c.) $|I| - 1 \leq \kappa(\Gamma_I(R))$.

Proof : (a) If $\Gamma(R/I)$ has only one vertex, then $\Gamma_I(R)$ consists of a single connected column and, therefore, is the complete graph on $|I|$ vertices.

(b) $1 \leq \kappa(\Gamma_I(R))$ because the graph is connected. Let $k = \kappa(\Gamma(R/I))$. Let $a_1+I, a_2+I, \ldots, a_k+I$ be vertices of $\Gamma(R/I)$ which, once removed, give a disconnected graph. Define G to be the graph obtained from $\Gamma_I(R)$ by removing the columns corresponding to $a_1 + I, \ldots, a_k + I$ (this means the removal of $k \cdot |I|$ vertices).

We show that G is disconnected. By our choice of $a_1+I, \ldots, a_k+I$, there exist vertices $b + I$ and $c + I$ of $\Gamma(R/I)$ such that $b + I$ is not connected to $c + I$ after $a_1 + I, \ldots, a_k + I$ are removed from $\Gamma(R/I)$. Then b and c are vertices of G. Suppose $b - x_1 - \ldots - x_m - c$ is a path in G. Without loss of generality, $x_j + I \neq x_{j+1} + I$ for $1 \leq j \leq m$ in view of Corollary 6. Therefore, $b + I - x_1 + I - \ldots - x_m + I - c + I$ is a path in $\Gamma(R/I)$ after $a_1 + I, \ldots, a_k + I$ have been removed. This is a contradiction. Hence G must be disconnected.

(c) Let $M = |I| - 1$ if $|I| < \infty$, and let M be any positive integer if $|I| = \infty$. Let $a_1, a_2, \ldots, a_M$ be any collection of vertices of $\Gamma_I(R)$. Define G to be the graph obtained by removing $a_1, \ldots, a_M$ from $\Gamma_I(R)$.

Let x and y be distinct vertices of G. We show that G is connected by finding a path from x to y in G. If x is adjacent to y, then we are done. Otherwise, recall that $diam(\Gamma_I(R)) \leq 3$ by Theorem 4. This implies the shortest path from x to y in $\Gamma_I(R)$ has length 2 or 3.

<u>*Case*1 :</u> $x - w - y$ is a path of shortest length from x to y in $\Gamma_I(R)$. If $w \neq a_j$ for any $j = 1, \ldots, M$, then this is also a path in G. So suppose $w = a_j$ for some j. The column $a_j + I$ contains $|I|$ elements, so we can choose $v \in a_j + I$ such that $v \neq a_i$ for any $i = 1, \ldots, M$.

Then x and y adjacent to w imply both vertices are also adjacent to v. Hence, $x - v - y$ is a path in G.

<u>*Case2 :*</u> $x - w - v - y$ is a path of shortest length from x to y in $\Gamma_I(R)$. Note that $w + I \neq v + I$ since, otherwise, this implies x is adjacent to v and so $x - v - y$ is a path in $\Gamma_I(R)$. Therefore, we can choose $a \in w+I$ and $b \in v+I$ such that a and b are vertices of G, i.e. $a, b \notin \{a_j | j = 1, \ldots, M\}$. Then x adjacent to w and y adjacent to v imply x is adjacent to a and y is adjacent to b. Also, w adjacent to v implies a is adjacent to b. Hence, $x - a - b - y$ is a path in G. □

COROLLARY 19. *If $I \neq (0)$ is a proper ideal but not a prime ideal, then $|I| - 1 \leq \kappa(\Gamma_I(R)) \leq (|I| \cdot \kappa(\Gamma(R/I)))$. In particular, $\kappa(\Gamma_I(R)) = \infty$ if I is infinite.*

Clique Number

For a graph G, a complete subgraph of G is called a *clique*. The *clique number*, $\omega(G)$, is the greatest integer $n \geq 1$ such that $K^n \subseteq G$, and $\omega(G) = \infty$ if $K^n \subseteq G$ for all $n \geq 1$. Note that $\omega(\Gamma(R/I)) \leq \omega(\Gamma_I(R))$ since $\Gamma(R/I)$ is isomorphic to a subgraph of $\Gamma_I(R)$. Equality holds if $\omega(\Gamma(R/I)) = \infty$. The next few results identify other cases in which equality holds.

PROPOSITION 20. *Let I be an ideal of the ring R. If $a + I$ is a connected column of $\Gamma_I(R)$, then $a+I$ is a complete subgraph of $\Gamma_I(R)$, and thus $\omega(\Gamma_I(R)) \geq |I|$.*

COROLLARY 21. *If $\Gamma_I(R)$ has at least one connected column and I is infinite, then $\omega(\Gamma_I(R)) = \infty$.*

COROLLARY 22. *If $\Gamma(R/I)$ consists of a single vertex, then $\omega(\Gamma_I(R)) = |I|$. Thus, if $I \neq (0)$, then $\omega(\Gamma(R/I)) < \omega(\Gamma_I(R))$.*

Proof : If $\Gamma(R/I)$ is a single vertex, then $\Gamma_I(R)$ consists of a single column. Since $\Gamma_I(R)$ must be connected, this is a connected column. □

COROLLARY 23. *If $\Gamma_I(R)$ has a connected column and $\Gamma(R/I)$ has at least two vertices, then $\omega(\Gamma_I(R)) \geq |I| + 1$.*

Proof : Let $a + I$ be a connected column of $\Gamma_I(R)$. By the hypothesis, there exists $b \in R - I$ such that $a + I \neq b + I$ and $a + I$ is adjacent to $b + I$ in $\Gamma(R/I)$. Then each element of the connected column $a + I$ is adjacent to b, and so $(a + I) \cup \{b\}$ forms a complete subgraph. □

Note that Figure 17 shows that it is necessary for $\Gamma(R/I)$ to have at least two vertices in Corollary 23.

THEOREM 24. *Let I be an ideal of a ring R. If $\Gamma_I(R)$ has no connected columns, then $\omega(\Gamma(R/I)) = \omega(\Gamma_I(R))$.*

FIGURE 17. $\Gamma_I(R)$, where $R = \mathbb{Z}_4 \times \mathbb{Z}_2$ and $I = 0 \times \mathbb{Z}_2$. This is an example where $1 = \omega(\Gamma(R/I)) < \omega(\Gamma_I(R)) = 2$.

Proof : We have already observed that $\omega(\Gamma(R/I)) \leq \omega(\Gamma_I(R))$. Thus, it is enough to consider the case where $\omega(\Gamma(R/I)) = N < \infty$. Assume G is a complete subgraph of $\Gamma_I(R)$ on the set of (distinct) vertices $a_1, a_2, \ldots, a_{N+1}$, and we provide a contradiction. Consider the subgraph G^* of $\Gamma(R/I)$on the vertices $a_1 + I, \ldots, a_{N+1} + I$. G^* is a complete subgraph of $\Gamma(R/I)$ since G is a complete subgraph of $\Gamma_I(R)$. But $\omega(\Gamma(R/I)) = N$ implies $a_j + I = a_k + I$ for some $j \neq k$. Write $a_j = a_k + i$ for some $i \in I$. Since G is complete, a_k is adjacent to a_j in $\Gamma_I(R)$. Thus $a_k^2 + ia_k = a_k(a_k + i) = a_k a_j \in I$. Hence $a_k^2 \in I$ and therefore $a_k + I$ is a connected column of $\Gamma_I(R)$. This is a contradiction. □

Girth

In this section we refine our results on the girth of $\Gamma_I(R)$. Recall, that if $\Gamma_I(R)$ contains a cycle, $gr(\Gamma_I(R)) \leq 7$ by Theorem 4. By Theorem 1.6 of [8] and as in (1.4) of [**16**], either $gr(\Gamma(R)) \leq 4$ or $gr(\Gamma(R)) = \infty$ for any ring R. We show that, given any ideal I of a ring R, the same result holds for $\Gamma_I(R)$.

LEMMA 25. *Let I be an ideal of a ring R. Then* $gr(\Gamma_I(R)) \leq gr(\Gamma(R/I))$.

Proof : If $gr(\Gamma(R/I)) = \infty$ we are done. So suppose $gr(\Gamma(R/I)) = n < \infty$. Let $x_1 + I - x_2 + I - \cdots - x_n + I - x_1 + I$ be a cycle in $\Gamma(R/I)$ through n distinct vertices. Then $x_1 - x_2 - \ldots - x_n - x_1$ is a cycle in $\Gamma_I(R)$ of length n. Hence, $gr(\Gamma_I(R)) \leq n$. □

LEMMA 26. *Let I be an ideal of a ring R. If $|I| \geq 3$ and $\Gamma_I(R)$ contains a connected column, then $gr(\Gamma_I(R)) = 3$.*

Proof : Let $x + I$ be a connected column of $\Gamma_I(R)$. Then $x^2 \in I$. Let $i, j \in I - \{0\}$. Then $x - (x+i) - (x+j) - x$ is a cycle of length 3 in $\Gamma_I(R)$. (Of course, $gr(G) \geq 3$ for any graph G.) □

LEMMA 27. *Let I be an ideal of a ring R. If $I \neq (0)$ and $\Gamma(R/I)$ has at least 2 vertices, then $gr(\Gamma_I(R)) \leq 4$.*

Proof : Let $x+I$ and $y+I$ be distinct adjacent vertices of $\Gamma(R/I)$. Then every element of $x + I$ is adjacent to every element of $y + I$ in $\Gamma_I(R)$. Let $0 \neq i \in I$. Then $x - (y+i) - (x+i) - y - x$ is a cycle of length 4 in $\Gamma_I(R)$. □

LEMMA 28. *Let I be an ideal of a ring R. If $I \neq (0)$ and $\Gamma(R/I)$ has only one vertex, then*
$$gr(\Gamma_I(R)) = \begin{cases} 3 & \text{if } |I| \geq 3, \\ \infty & \text{if } |I| = 2. \end{cases}$$

Proof : If $\Gamma(R/I)$ has only one vertex, then $\Gamma_I(R)$ consists of a single connected column. Thus $\Gamma_I(R)$ is a complete graph and, therefore has a cycle of length 3 unless $\Gamma_I(R)$ has only two vertices. □

LEMMA 29. *Let I be an ideal of a ring R that is not prime. If $I \neq (0)$, $\Gamma_I(R)$ has no connected columns, and $gr(\Gamma(R/I)) > 3$, then $gr(\Gamma_I(R)) = 4$.*

Proof : Since $\Gamma_I(R)$ has no connected columns, $\Gamma(R/I)$ must have at least two vertices. By Lemma 26, $gr(\Gamma_I(R)) \leq 4$. Assume $x-y-z-x$

is a cycle in $\Gamma_I(R)$ of length 3 and we provide a contradiction. Since $gr(\Gamma(R/I)) > 3$, $x+I-y+I-z+I-x+I$ cannot be a cycle in $\Gamma(R/I))$. Therefore, we have either $x+I=y+I$, $y+I=z+I$, or $z+I=x+I$. If $x+I=y+I$, then $(x+I)^2=(x+I)(y+I)=0+I$ and so $x+I$ is a connected column of $\Gamma_I(R)$. But this is a contradiction. We get a similar contradiction if $y+I=z+I$ or $z+I=x+I$. Hence, $gr(\Gamma_I(R))=4$. □

LEMMA 30. *Let I be an ideal of a ring R. If I has two elements, $\Gamma(R/I)$ has at least two vertices, and $\Gamma_I(R)$ has at least one connected column, then $gr(\Gamma_I(R))=3$.*

Proof : Let $x+I$ be a connected column of $\Gamma_I(R)$. Then $x^2 \in I$. Let $y+I$ be a vertex adjacent to $x+I$ in $\Gamma(R/I)$. Write $I=\{0,i\}$. Then $y-x-x+i-y$ is a cycle of length 3 in $\Gamma_I(R)$. □

The next theorem summarizes these results.

THEOREM 31. *Let I be a nonzero ideal of a ring R that is not a prime ideal. Then*

$$gr(\Gamma_I(R)) = \begin{cases} \infty & \text{if } \Gamma(R/I) \text{ has only one vertex and } |I|=2, \\ 4 & \text{if } gr(\Gamma(R/I))>3 \text{ and} \\ & \quad \Gamma_I(R) \text{ has no connected columns,} \\ 3 & \text{otherwise} \end{cases}$$

COROLLARY 32. *Let I be a nonzero ideal of a ring R. If either (a.) $\Gamma_I(R)$ is the connected graph on two vertices, or (b.) $gr(\Gamma(R/I))>3$, $\Gamma(R/I)$ has at least two vertices, and $\Gamma_I(R)$ has no connected columns, then $\omega(\Gamma_I(R))=2$. In all other cases, $\omega(\Gamma_I(R))\geq 3$.*

Proof : A cycle of length 3 is a complete graph on 3 vertices. Hence, if $gr(\Gamma_I(R)) \neq 3$, then there can be no complete subgraphs of $\Gamma_I(R)$ on three or more vertices. □

Figure 17 gives an example where $gr(\Gamma_I(R)) = \infty$. Figure 15 gives an example where $gr(\Gamma_I(R)) = 3$. Figure 14 illustrates condition (b) in the above corollary and has girth 4. Figure 18 is another example with $gr(\Gamma_I(R)) = 4$.

A graph G is *bipartite* with vertex classes V_1, V_2 if the set of all vertices of G is $V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and each edge of G joins a vertex from V_1 to a vertex of V_2. A *complete bipartite graph* is a bipartite graph containing all edges joining the vertices of V_1 and V_2.

THEOREM 33. *Let I be a nonzero ideal of a ring R. Then $\Gamma_I(R)$ is bipartite if and only if either (a.) $gr(\Gamma_I(R)) = \infty$ or (b.) $gr(\Gamma_I(R)) = 4$ and $\Gamma(R/I)$ is bipartite.*

Proof : Suppose that $\Gamma_I(R)$ is bipartite. Since $\Gamma(R/I)$ is isomorphic to a subgraph or $\Gamma_I(R)$, $\Gamma(R/I)$ is bipartite (or a single vertex).

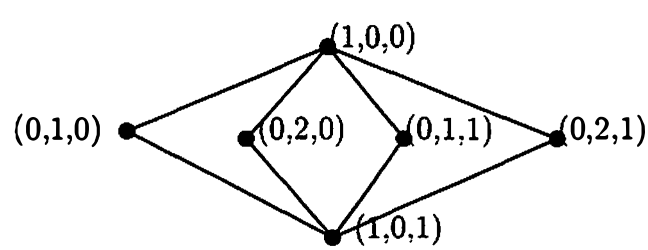

FIGURE 18. $\Gamma_I(R)$, where $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ and $I = 0 \times 0 \times \mathbb{Z}_2$.

By Theorem 31, $gr(\Gamma_I(R))$ is 3, 4, or ∞. By Theorem 1 of section 1.2 of [6], a graph is bipartite if and only if it does not contain an odd cycle. Hence $gr(\Gamma_I(R)) \neq 3$.

If $gr(\Gamma_I(R)) = \infty$, then by Theorem 31 $\Gamma_I(R)$ is a graph on two vertices and therefore bipartite. Suppose $gr(\Gamma_I(R)) = 4$ and $\Gamma(R/I)$ is bipartite. Let W_1, W_2 be the two vertex classes of $\Gamma(R/I)$. Define $V_j = \{x + i | i \in I,\ x + I \in W_j\}$ for $j = 1, 2$. Then $V_1 \cap V_2 = \emptyset$ and the vertex set of $\Gamma_I(R)$ is $V_1 \cup V_2$. Let x and y be adjacent vertices of $\Gamma_I(R)$. Without loss of generality, say $x \in V_1$. By Theorem 31, $\Gamma_I(R)$ has no connected columns. Thus $x + I \neq y + I$. Hence, $x + I - y + I$ is an edge in $\Gamma(R/I)$ by Theorem 5. Since $x + I \in W_1$, $y + I \in W_2$. Therefore $y \in V_2$. Hence, all edges of $\Gamma_I(R)$ join vertices from V_1 to those of V_2. Thus $\Gamma_I(R)$ is bipartite. □

COROLLARY 34. *Let I be a nonzero ideal of a ring R. Then $\Gamma_I(R)$ is complete bipartite if and only if either (a.) $gr(\Gamma_I(R)) = \infty$ or (b.) $gr(\Gamma_I(R)) = 4$ and $\Gamma(R/I)$ is complete bipartite.*

Proof : If $\Gamma_I(R)$ is complete bipartite, then the result follows as above.

If $gr(\Gamma_I(R)) = \infty$, then again the result is trivial. So suppose $gr(\Gamma_I(R)) = 4$ and $\Gamma(R/I)$ is complete bipartite. Define W_j, V_j as above. Let $a \in V_1$ and $b \in V_2$. Then $a + I \in W_1$ and $b + I \in W_2$. Since $\Gamma(R/I)$ is complete bipartite, $a+I$ is adjacent to $b+I$ in $\Gamma(R/I)$. Thus, a is adjacent to b in $\Gamma_I(R)$ by Theorem 5. □

Graphs on n Vertices

We now turn to investigating which graphs on n vertices are achievable as $\Gamma_I(R)$ for some ring R and ideal I of R for specific values of n. By Lemma 10, $\Gamma_I(R)$ can be a graph on one vertex only if $I = (0)$ and $\Gamma(R)$ is a graph on one vertex (for example, $R = \mathbb{Z}_4$). There is only one connected graph on two vertices, and Figure 17 shows this is achievable as $\Gamma_I(R)$ with $I \neq (0)$. This graph is also achievable with $I = (0)$ and $R = \mathbb{Z}_9$.

There are two connected graphs on 3 vertices, as shown in Figure 19.

Graph A is achievable if $R = \mathbb{Z}[X,Y]/(X^2, XY, Y^2)$ and $I = (0)$. Graph B is achievable if $R = \mathbb{Z}_6$ or $\mathbb{Z}_8$ and $I = (0)$. We show Graph A is achievable with $I \neq (0)$, but Graph B is not.

THEOREM 35. *Let $n \geq 2$ be an integer. Then there is a ring R and with a nonzero ideal I such that $K^n \simeq \Gamma_I(R)$.*

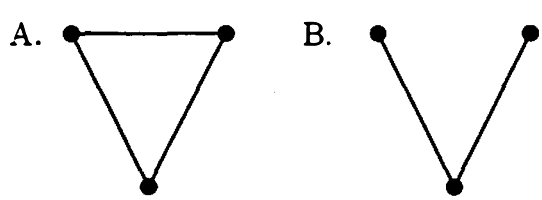

FIGURE 19. The connected graphs on 3 vertices.

Proof : Let $R = \mathbb{Z}_4 \times \mathbb{Z}_n$ and let $I = 0 \times \mathbb{Z}_n$. Then $R/I \simeq \mathbb{Z}_4$, and so $\Gamma(R/I)$ is a graph on one vertex. Thus $\Gamma_I(R)$ consists of one connected column of height n. □

PROPOSITION 36. *Graph B in Figure 19 is not achievable as* $\Gamma_I(R)$ *for any ring* R *and nonzero ideal* I *of* R.

Proof : If Graph B is achievable, then $|I| = 3$ and $\Gamma(R/I) = 1$ by Lemma 10 since $I \neq (0)$. Thus $\Gamma_I(R)$ consists of a single connected column. But this implies $\Gamma_I(R)$ is a complete graph, and Graph B is not complete. □

There are six connected graphs on 4 vertices, as shown in Figure 20.

Graphs II, III, and IV are the only graphs achievable as $\Gamma_I(R)$ with $I = (0)$ as shown in Example 2.1 of [2]. By Theorem 35, Graph III is achievable with $I \neq (0)$. We show that the only other graph achievable with $I \neq (0)$ is graph II, as shown in Figure 21.

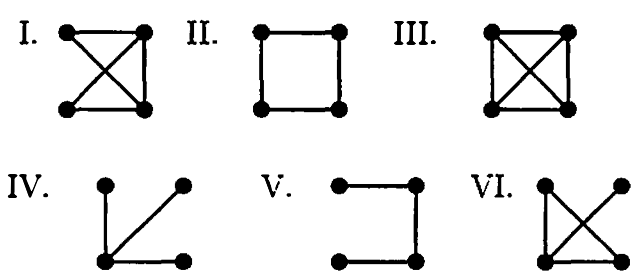

FIGURE 20. The connected graphs on 4 vertices.

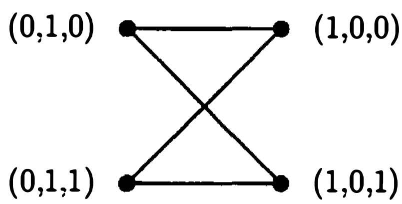

FIGURE 21. $\Gamma_I(R)$, where $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = 0 \times 0 \times \mathbb{Z}_2$. Note that $\Gamma_I(R)$ is isomorphic to graph II of Figure 20.

THEOREM 37. *Let I be a nonzero ideal of a ring R. Then, unless $\Gamma_I(R)$ is the graph on two vertices, no vertex of $\Gamma_I(R)$ is adjacent to only one other vertex.*

Proof : If $\Gamma(R/I)$ is a single vertex, then $\Gamma_I(R)$ is a single connected column and therefore a complete graph. Hence, each vertex is adjacent to $|I| - 1$ other vertices. Note that, in this case, $\Gamma_I(R)$ is the graph on two vertices only if $|I| = 2$.

So suppose that $\Gamma(R/I)$ has at least 2 vertices. Let x be any vertex of $\Gamma_I(R)$. Let $y + I$ be any vertex adjacent to $x + I$ in $\Gamma(R/I)$, and let $0 \neq i \in I$. Then x is adjacent to both y and $y + i$ in $\Gamma_I(R)$. □

COROLLARY 38. *Graphs IV, V, and VI in Figure 20 are not achievable as $\Gamma_I(R)$ for any ring R and nonzero ideal I of R.*

LEMMA 39. *Let I be an ideal of a ring R. If $\Gamma(R/I)$ is a graph on two vertices, then $\Gamma_I(R)$ has either no or exactly two connected columns.*

Proof : Let x and y be the vertices of $\Gamma(R/I)$. Then, in R/I, $xy = 0$, $x \neq 0$, $y \neq 0$, and the entire set of zero-divisors of R/I is $\{x, y, 0\}$. Note that $x^2y = x(xy) = x0 = 0$. So $x^2 \in \{x, y, 0\}$.

If $x^2 = y$, then $x^3 = x(x^2) = xy = 0$. Thus $y(y-x) = x^2(x^2-x) = x^4 - x^3 = 0$. Hence $y - x$ is a zero-divisor. But $y - x \neq 0$ since $x \neq y$, $y - x \neq x$ since $y \neq 0$, and $y - x \neq y$ since $x \neq 0$. This is a contradiction. Hence, either $x^2 = x$ or $x^2 = 0$.

Suppose that $x^2 = x$. Then x is not a connected column of $\Gamma_I(R)$. Assume that y is a connected column of $\Gamma_I(R)$; that is, assume $y^2 = 0$. Then $y(y-x) = y^2 - xy = 0$. Thus $y - x$ is a zero-divisor of R/I. But, as above, this gives a contradiction. Hence, in this case, neither x nor y are connected columns of $\Gamma_I(R)$.

Suppose that $x^2 = 0$. Then x is a connected column of $\Gamma_I(R)$. Assume y is not a connected column of $\Gamma_I(R)$; that is, assume $y^2 \neq 0$. Then, as in the second paragraph, we must have $y^2 = y$. But, by an argument paralleling that of the previous paragraph, we now get a contradiction. Hence, both x and y are connected columns of $\Gamma_I(R)$. □

PROPOSITION 40. *Graph I in Figure 20 is not achievable as $\Gamma_I(R)$ for any ring R and nonzero ideal I of R.*

Proof : Assume there is a ring R with nonzero ideal I such that $\Gamma_I(R)$ is isomorphic to graph I. By Lemma 10, $4 = N \cdot |I|$ where N is the number of vertices of $\Gamma(R/I)$. Note that $|I| \neq 4$, since otherwise $\Gamma_I(R)$ is a single connected column and therefore complete. Hence, because $I \neq (0)$, we must have $|I| = 2$ and $\Gamma(R/I)$ is a graph on two vertices.

By the above lemma, $\Gamma_I(R)$ has either no or exactly two connected columns. However, if $\Gamma_I(R)$ has two connected columns, then $\Gamma_I(R)$ is isomorphic to K^4. If $\Gamma_I(R)$ has no connected columns, then $\Gamma_I(R)$ is isomorphic to the graph in Figure 21 (which also has zero connected columns, $|I| = 2$, and $\Gamma(R/I)$ as the graph on two vertices). $\square$

We now give a partial answer as to which graphs on p vertices are achievable as $\Gamma_I(R)$ for some ring R and nonzero I ideal of R where p is prime. It is an open question as to which graphs on p vertices are achievable as $\Gamma(R)$ for some ring R for $p \geq 5$.

PROPOSITION 41. *Let p be a prime integer. Then the only graphs on p vertices achievable as $\Gamma_I(R)$ for some ring R and ideal I of R are K^p and those graphs achievable as $\Gamma(S)$ for some ring S. If we require $I \neq (0)$, then the only achievable graph is K^p.*

Proof : By Lemma 10, we must have either $|I| = p$ and $\Gamma(R/I)$ is the graph with one vertex, or $|I| = 1$ and $\Gamma(R/I)$ is a graph with p vertices. That is, either $\Gamma_I(R)$ is a single connected column, and therefore complete, or $I = (0)$. If $I = (0)$, then $\Gamma_I(R) \simeq \Gamma(R)$. $\square$

Next, let us consider which graphs on 6 vertices are achievable as $\Gamma_I(R)$ for some ring R and ideal I of R. If $I = (0)$, then a graph can be achieved as $\Gamma_I(R)$ if and only if it can be achieved as $\Gamma(R)$. If I is nonzero, then either $\Gamma(R/I)$ is a graph on one, two, or three vertices. In the first case, the graph is isomorphic to K^6. If $\Gamma(R/I)$ is a graph on two vertices, then $\Gamma_I(R)$ has either no or exactly two connected columns by Lemma 39. Therefore, our graph is isomorphic to one of the graphs in Figure 22.

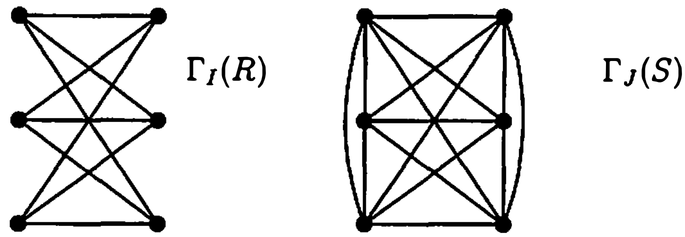

FIGURE 22. $\Gamma_I(R)$, where $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and $I = 0 \times 0 \times \mathbb{Z}_3$; and $\Gamma_J(S)$, where $S = \mathbb{Z}_9 \times \mathbb{Z}_3$ and $J = 0 \times \mathbb{Z}_3$. (Note that $\Gamma_J(S) \simeq K^6$.)

For the case where $\Gamma(R/I)$ is the graph on three vertices, we need the following lemmas.

LEMMA 42. *Let I be an ideal of a ring R. If $\Gamma(R/I)$ is the complete graph on three vertices, then $\Gamma_I(R)$ has three connected columns.*

Proof : Let x, y, and z be the vertices of $\Gamma(R/I)$. That is, $x, y, z \in R/I, xy = yz = xz = 0, x \neq 0, y \neq 0, z \neq 0$ and the only zero-divisors of R/I are x, y, z, and 0. Note that $(x+y)z = xz + yz = 0$. Therefore, $x + y$ is a zero-divisor of R/I. Clearly, $x + y \neq x$ and $x + y \neq y$.

<u>*Case 1* :</u> Suppose $x+y = z$. Then $x^2 = x^2+xy = x(x+y) = xz = 0$, and similarly $y^2 = 0$. Thus, $z^2 = (x+y)^2 = x^2 + 2xy + y^2 = 0$.

<u>*Case 2* :</u> Suppose $x + y = 0$, that is $x = -y$. Then $x^2 = x(-y) = -xy = 0$, and similarly $y^2 = 0$. Note that $(x+z)y = xy + zy = 0$, and therefore $x + z$ is a zero-divisor of R/I. Clearly, $x + z \neq x$ and $x + z \neq z$. If $x + z = y$, then $z^2 = 0$ as in Case 1. If $x + z = 0$, then $z^2 = 0$ as in Case 2.

Thus, in all cases, $x^2 = y^2 = z^2 = 0$. □

Note that if I is an ideal of a ring R satisfying the above hypothesis and such that $|I| = 2$, each vertex of $\Gamma_I(R)$ is adjacent to every other vertex by Theorem 5. Hence, in this case, $\Gamma_I(R) \simeq K^6$.

LEMMA 43. *Let I be an ideal of a ring R. If $\Gamma(R/I)$ is the graph with three vertices and two edges, then $\Gamma_I(R)$ has either no connected columns, or the column represented by the element adjacent to two vertices in $\Gamma(R/I)$ is the only connected column.*

Proof : By hypothesis, the set of zero-divisors of R/I is $\{x, y, z, 0\}$, where, without loss of generality, $xy = yz = 0$ and $xz \neq 0$. It suffices to show $x^2 \neq 0$ and $z^2 \neq 0$. Note that $(x + z)y = xy + zy = 0$. Thus $x + z$ is a zero-divisor of R/I. Clearly, $x + z \neq x$ and $x + z \neq z$.

<u>*Case* 1 :</u> Suppose $x + z = 0$. That is, $x = -z$. Then $x^2 = x(-z) = -xz \neq 0$, and similarly $z^2 \neq 0$.

<u>*Case* 2 :</u> Suppose $x + z = y$. Then $y^2 = (x + z)y = 0$. Now $0 = xy = y^2 + xy - y^2 = (y+x)y - y^2 = (x+z+x)(x+z) - (x+z)^2 = 2x^2 + 3xz + z^2 - x^2 - 2xz - z^2 = x^2 + xz$. Thus, $x^2 = -xz \neq 0$. A similar argument shows $z^2 \neq 0$. □

Figure 23 gives the two remaining graphs on six vertices that can be achieved as $\Gamma_I(R)$ with $I \neq (0)$.

Now, we consider which graphs on nine vertices are achievable as $\Gamma_I(R)$ for some ring R and ideal I of R. If $I = (0)$, then a graph can be achieved as $\Gamma_I(R)$ if and only if it can be achieved as $\Gamma(R)$. If $\Gamma(R/I)$ is a graph on one vertex, then $\Gamma_I(R) \simeq K^9$. Otherwise, $\Gamma(R/I)$ is a graph on three vertices and $|I| = 3$. If $\Gamma(R/I)$ is complete, then again $\Gamma_I(R) \simeq K^9$ by Theorem 5 and and Lemma 10. If $\Gamma(R/I)$ is

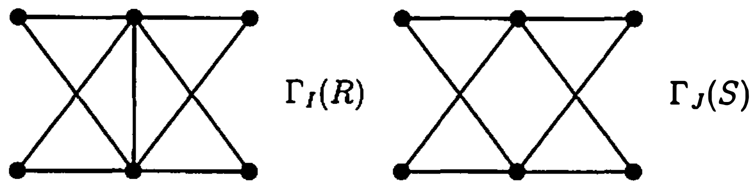

FIGURE 23. $\Gamma_I(R)$, where $R = \mathbb{Z}_2[x]/(x^3) \times \mathbb{Z}_2$ and $I = 0 \times \mathbb{Z}_2$; and $\Gamma_J(S)$, where $S = \mathbb{Z}_6 \times \mathbb{Z}_2$ and $J = 0 \times \mathbb{Z}_2$. (Note that we have a different structure here because S/J is reduced and R/I is not.)

not complete, then, by Lemma 43, the only possibilities are graphs isomorphic to Figures 14 or 15.

Finally, we analyze which graphs on eight vertices are achievable as $\Gamma_I(R)$ for some ring R and ideal I of R. As in the previous cases, for $I = (0)$, a graph can be achieved as $\Gamma_I(R)$ if and only if it can be achieved as $\Gamma(R)$; and, if $\Gamma(R/I)$ consists of a single vertex, then $\Gamma_I(R) \simeq K^8$.

By Lemma 10, there are two other cases to consider: $\Gamma(R/I)$ consists of two or four vertices. In the first case, the graph has either no or exactly two connected columns by Lemma 39. If $\Gamma_I(R)$ has two connected columns, then $\Gamma_I(R) \simeq K^8$. Figure 24 illustrates the other case.

If $\Gamma(R/I)$ is a graph on four vertices, there are three possibilities: Graphs II, III, and IV from Figure 20. We examine each of these possibilities for connected columns of $\Gamma_I(R)$ after proving a lemma. Recall that the *degree* of a vertex y in a graph G, denoted $deg(y)$, is

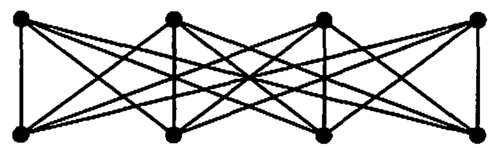

FIGURE 24. $\Gamma_I(R)$, where $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ and $I = 0 \times 0 \times \mathbb{Z}_4$. (Here, for convenience, we have placed the columns horizontally.)

the number of vertices of G which are adjacent to y (not including y itself, of course).

LEMMA 44. *Let R be a ring such that $\Gamma(R)$ has at least three vertices. If $0 \neq x \in R$ such that $x^2 = 0$, then $deg(x) \geq 2$.*

Proof : Since x is a zero-divisor of R, it is a vertex of $\Gamma(R)$. Because $\Gamma(R)$ is connected, there is some $0 \neq y \in R$ such that $y \neq x$ and $xy = 0$.

We assume $deg(x) = 1$ and provide a contradiction. By our assumption, if $a \in R$ with $ax = 0$, then $a \in \{x, y, 0\}$. Note that $(x + y)x = x^2 + xy = 0$. Clearly, $x + y \neq x$ and $x + y \neq y$. Thus $x + y = 0$; that is, $y = -x$. Since $deg(x) = 1$, $\Gamma(R)$ is connected, and $\Gamma(R)$ has at least three vertices, y must be adjacent to another vertex. Therefore, there exists some $0 \neq z \in R$ such that $z \neq y$, $z \neq x$, and $yz = 0$. But then, $xz = (-y)z = -yz = 0$. This contradicts $deg(x) = 1$. $\square$

PROPOSITION 45. *Let I be an ideal of a ring R such that $\Gamma(R/I)$ is a graph on four vertices isomorphic to Graph IV of Figure 20. Then $\Gamma_I(R)$ has no connected columns.*

Proof : Let $\{a, b, c, d\}$ be the vertices of $\Gamma(R/I)$ where $deg(a) = deg(b) = deg(d) = 1$ and $deg(c) = 3$. By the preceding lemma, $a^2 \neq 0$, $b^2 \neq 0$, and $d^2 \neq 0$. Thus, it is enough to show $c^2 \neq 0$.

We assume $c^2 = 0$ and provide a contradiction. Note that $(a+c)c = ac + c^2 = 0$. Thus $a + c$ is a zero-divisor of R/I. Clearly, $a + c \neq a$ and $a + c \neq c$. Also, $a + c \neq 0$ (otherwise, $0 = c^2 = (-a)^2 = a^2 \neq 0$). So, without loss of generality, $a + c = b$.

Now $(b + c)c = bc + c^2 = 0$, and so $b + c$ is a zero-divisor of R/I. Clearly $b + c \neq b$, $b + c \neq c$, and (as above) $b + c \neq 0$. If $a = b + c$, then $a = (a + c) + c$ and thus $2c = 0$. Consider $d + c$, which is a zero-divisor of R/I with $d + c \neq d, c, 0$ as above. Note that $d + c \neq b = a + c$ since $d \neq a$ and $d + c \neq a = b + c$ since $d \neq b$. But this exhausts all the possible zero-divisors of R/I, contradicting $b + c = a$. Thus, we must conclude $b + c = d$.

Again consider $d + c$, which is a zero-divisor of R/I with $d + c \neq d, c, 0$. Note that $d + c \neq b = a + c$ since $d \neq a$. Therefore, we must have $d + c = a$. Then $a = d + c = (b + c) + c = [(a + c) + c] + c = a + 3c$. So $3c = 0$. Now, since $c \neq 0$, $-c \neq c$, and clearly $-c$ is a zero-divisor of R/I. Thus $-c \in \{a, b, d\}$. But $(-c)^2 = c^2 = 0$, and, by the above lemma, $a^2 \neq 0, b^2 \neq 0$, and $d^2 \neq 0$. This contradicts $b + c = d$.

Now we have exhausted all the possible zero-divisors of R/I as $b + c$. We get a similar contradiction if we suppose $a + c = d$. Hence, we cannot have $c^2 = 0$. □

PROPOSITION 46. *Let I be an ideal of a ring R such that $\Gamma(R/I)$ is the complete graph on four vertices (as in Graph III of Figure 20). Then $\Gamma_I(R)$ consists of four connected columns.*

Proof : Let $a, b, c, d \in R/I$ be the vertices of $\Gamma(R/I)$. Since $\Gamma(R/I)$ is a complete graph, the product of any two distinct zero-divisors of R/I is zero. It is enough to show that each of these elements represents a connected column of $\Gamma_I(R)$.

We assume $a^2 \neq 0$ and provide a contradiction. Note that $(a+b)d = ad + bd = 0$ and $(a + c)d = ad + cd = 0$. Therefore, $a + b$ and $a + c$ are distinct zero-divisors of R/I, but $(a + b)(a + c) = a^2 + ab + ac + bc = a^2 \neq 0$. This is a contradiction. The proof that $b^2 = c^2 = d^2 = 0$ is similar. □

COROLLARY 47. *If, in addition to the above hypothesis, $\Gamma_I(R)$ is a graph on eight vertices, then $\Gamma_I(R) \simeq K^8$.*

Figure 25 gives an example of such a ring R and ideal I of R as in Proposition 45 and the resulting ideal-based graph.

PROPOSITION 48. *Let I be an ideal of a ring R such that $\Gamma(R/I)$ is isomorphic to Graph II of Figure 20. Then $\Gamma_I(R)$ has no connected columns.*

Proof : Let a be a vertex of $\Gamma(R/I)$. Then a is adjacent to two distinct vertices, say b and c, of $\Gamma(R/I)$, and there is another vertex,

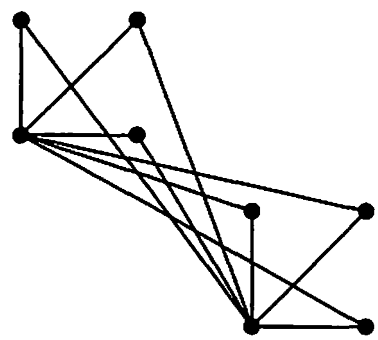

FIGURE 25. $\Gamma_I(R)$, where $R = \mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{Z}_2$ and $I = 0 \times 0 \times \mathbb{Z}_2$.

say d, of $\Gamma(R/I)$ that is not adjacent to a. Note that d is adjacent to both b and c.

We assume $a^2 = 0$ and provide a contradiction. Note that $(a + b)a = a^2 + ab = 0$. Thus $a + b$ is an annihilator of a in R/I. So $a + b \in \{a, b, c, 0\}$. Clearly $a + b \neq a$ and $a + b \neq b$. If $a + b = 0$, then $0 = bd = (-a)d = -(ad) \neq 0$. If $a + b = c$, then $0 = cd - bd = (c - b)d = ad \neq 0$. Thus we have exhausted all possibilities for $a + b$ as an annihilator of a. This contradicts $a^2 = 0$. □

Figure 26 gives an example of a ring R and ideal I as in the above Proposition.

In summary, the above work completely determines which graphs on n vertices can be realized as $\Gamma_I(R)$ for some ring R and ideal I of R for $n = 1, 2, 3, 4$. If we require I to be nonzero, then we also have a complete determination for $n = 6, 8, 9$ and any prime $n \geq 5$. If we knew

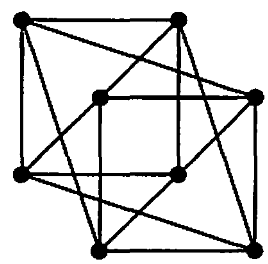

FIGURE 26. $\Gamma_I(R)$, where $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ and $I = 0 \times 0 \times \mathbb{Z}_2$.

which graphs on n vertices could be realized as $\Gamma(R)$ for these values of n, then again we have a complete determination of possibilities for $\Gamma_I(R)$ for any ideal I. Currently, this is an open question.

Connected and Nonconnected Type

DEFINITION 49. *Let R be a ring. We define $\gamma(R) = \{\Gamma_I(R) | I$ is a nonzero proper ideal of $R\}$.*

(a.) A ring R is of connected type if $\gamma(R) \neq \{\emptyset\}$ and each nonempty element of the collection $\gamma(R)$ has at least one connected column.

(b.) A ring R is of nonconnected type if $\gamma(R) \neq \{\emptyset\}$ and no element of $\gamma(R)$ has a connected column.

(c.) A ring R is of mixed type if it is not of connected or nonconnected type.

EXAMPLE 50. 1. The ring $R = \mathbb{Z}_8$ is of connected type. Note that R has only two nontrivial ideals, $I = 2R$ and $J = 4R$. It is easy to verify that $\Gamma_I(R)$ is the empty graph and $\Gamma_J(R)$ consists of one connected column.

2. Note that $\Gamma_{8\mathbb{Z}}(\mathbb{Z})$ contains a connected column represented by $4+8\mathbb{Z}$, and that $\Gamma_{6\mathbb{Z}}(\mathbb{Z})$ consists of three nonconnected columns: $2+6\mathbb{Z}, 3+6\mathbb{Z}$, and $4+6\mathbb{Z}$. Thus $\mathbb{Z}$ is of mixed type.

We can reformulate these types in terms of purely ring-theoretic properties of R. Recall that for an ideal I of a ring R, we define the *radical* of I, $Rad(I) = \{x \in I | x^n \in I \text{ for some integer } n \geq 1\}$. $Rad(I)$ is an ideal of R containing I for any ring R. An ideal I of R is called a *radical ideal* if $Rad(I) = I$.

THEOREM 51. *Let R be a ring. Then R is of connected type if and only if (a.) for every nonzero proper ideal I of R, either $Rad(I) \neq I$ or R/I is an integral domain, and (b.) $Rad(J) \neq J$ for some nonzero proper ideal J of R.*

Proof : Suppose that R is of connected type. Let I be a nontrivial ideal of R such that $\Gamma_I(R) \neq \emptyset$ (some such I must exist since R is of connected type). Then there is a connected column of $\Gamma_I(R)$. Let $x + I$ represent this column. Then $x \in R - I$ and $x^2 \in I$. Thus $x \in Rad(I) - I$.

Conversely, suppose that conditions (a) and (b) of the hypothesis hold. Note that condition (b) implies $\gamma(R) \neq \{\emptyset\}$. So let I be a nontrivial ideal of R such that $\Gamma_I(R) \neq \emptyset$. Then R/I is not an integral domain. Therefore, by condition (a), $Rad(I) \neq I$. So, there is some $y \in Rad(I) - I$. There is a smallest integer $n \geq 2$ such that $y^n \in I$.

Then, $y^{n-1} \in R - I$ and $(y^{n-1})^2 \in I$. Therefore, $y^{n-1} + I$ represents a connected column of R. □

THEOREM 52. *Let R be a ring. Then R is of nonconnected type if and only if (a.) for every nonzero proper ideal I of R, Rad(I) = I and (b.)* $\gamma(R) \neq \{\emptyset\}$.

Proof : Suppose that R is of nonconnected type and assume there is a nontrivial ideal I of R such that $Rad(I) \neq I$. Let $x \in Rad(I) - I$. There is a smallest integer $n \geq 2$ such that $x^n \in I$. Then $x^{n-1} \in R - I$, and so $x^{n-1} + I$ represents a column of $\Gamma_I(R)$. But $(x^{n-1})^2 \in I$ implies this column is connected. This is a contradiction. Thus (a) and (b) hold.

Conversely, suppose that conditions (a) and (b) of the hypothesis hold. Let I be a nontrivial ideal of R. Then for each $x \in R - I$, $x^2 \notin I$ implies $(x + I)^2 \neq 0 + I$. Thus, $\Gamma_I(R)$ has no connected columns. □

This condition can be restated in more familiar terms. Given a ring R, let $nil(R)$ be the ideal of nilpotent elements of R. Note that $nil(R) = Rad(0)$. If R is reduced, then $nil(R) = (0)$.

PROPOSITION 53. *Let R be a ring such that* $nil(R) \neq (0)$ *and each nonzero ideal of R is a radical ideal. Then* $nil(R)$ *is principal and is the unique nonzero ideal of R. Moreover,* $(nil(R))^2 = (0)$ *and* $\Gamma(R)$ *is the complete graph on* $card(nil(R)) - 1$ *vertices.*

Proof : Let $0 \neq x \in nil(R)$. Then clearly $(x) \subseteq nil(R)$. If $y \in nil(R)$, then $y^n = 0 \in (x)$ for some positive integer n. Since (x) is a radical ideal, $y \in (x)$. Thus, $nil(R) \subseteq (x)$. Therefore, $nil(R)$ is a principal ideal.

For any $0 \neq y \in nil(R)$, we have shown $(y) = nil(R)$. We may choose $0 \neq x \in nil(R)$ such that $x^2 = 0$. Then $(y) = nil(R) = (x)$. Hence, we have $(y)^2 = (x)^2 = (0)$. Thus $y^2 = 0$, and therefore $(nil(R))^2 = (0)$.

Assume there is some $r \in R^*$ such that r is not nilpotent or a unit of R, and we provide a contradiction. By assumption, $r^2 \neq 0$. Since (r^2) is radical, $r \in (r^2)$. So $r = r^2a$ for some $a \in R$. Let $e = ra$. Then $e^2 = r^2a^2 = (r^2a)a = ra = e$. That is, e is idempotent. Also, $(e) = (r)$ because $e = ra$ and $r = r^2a = re$. Note then that $e \neq 0$. Since r is not a unit, $e \neq 1$. Thus $1 - e$ is also a nonzero idempotent of R.

Choose $0 \neq z \in nil(R)$. Then $z^2 = 0 \in (e)$. Since (e) is a radical ideal, $z \in (e)$. Write $z = \alpha e$ for some $\alpha \in R$. Similarly, $z \in (1-e)$. So $z = \beta(1-e)$ for some $\beta \in R$. Then $z = \alpha e = (\alpha e)e = (\beta(1-e))e = 0$. This contradicts our choice of z. Hence, each element of R is either nilpotent or a unit. Thus $nil(R)$ is the unique maximal ideal. The first paragraph of the proof shows that this must be the only nonzero proper ideal of R.

Note that the above paragraph shows that $Z(R) \subseteq nil(R) = (x)$, for some $0 \neq x \in nil(R)$ with $x^2 = 0$. But clearly $nil(R) \subseteq Z(R)$. Hence, the vertices of $\Gamma(R)$ are the nonzero elements of (x). Since $x^2 = 0$, all distinct vertices of $\Gamma(R)$ are adjacent. □

$\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p[X]/(X^2)$ for any prime p are examples of rings that satisfy the hypothesis of Proposition 53.

COROLLARY 54. *Let R be a ring. Then R is of nonconnected type if and only if R is von Neumann regular and* $\gamma(R) \neq \{\emptyset\}$.

Proof : Suppose that R is von Neumann regular and $\gamma(R) \neq \{\emptyset\}$. Thus, each ideal of R is a radical ideal. Then R is of nonconnected type by Theorem 52.

Conversely, suppose R is of nonconnected type. Then, by definition, $\gamma(R) \neq \{\emptyset\}$. Assume R is not von Neumann regular. By Theorem 52, the only possibility is that (0) is not a radical ideal. That is, R is a ring such that $nil(R) \neq (0)$ and every nonzero ideal of R is a radical ideal. But, by Proposition 53, $nil(R) = (x)$ is the unique nonzero ideal of R. Thus, in particular, it is a maximal ideal. Therefore, by Proposition 2, $\Gamma_{(x)}(R) = \emptyset$. Thus, $\gamma(R) = \{\emptyset\}$. This is a contradiction. Hence, R is von Neumann regular. □

THEOREM 55. *Let R be a ring.*

(a.) $\Gamma_I(R) = \emptyset$ *for every ideal I of R if and only if R is a field.*

(b.) If R is a reduced ring, then $\gamma(R) = \{\emptyset\}$ *if and only if either R is a field or* $R = K_1 \times K_2$ *for fields* K_1 *and* K_2.

(c.) If R is not a reduced ring, then $\gamma(R) = \{\emptyset\}$ *if and only if R is a quasilocal ring with unique maximal ideal* (x) *such that* $(x)^2 = (0)$.

Proof : (a) If R is a field, the result is clear.

Conversely, suppose that $\Gamma_I(R) = \emptyset$ for every ideal I of R. In particular, this implies each ideal of R is a radical ideal. Thus R is von Neumann regular. Since $\Gamma_{(0)}(R) = \Gamma(R) = \emptyset$, R is a domain. Hence, R must be a field.

(b) If R is a field, the result is clear. If $R = K_1 \times K_2$, the only nonzero ideals of R are maximal. Thus the result follows from Proposition 2.

Conversely, suppose that $\gamma(R) = \{\emptyset\}$. Since R is reduced, every ideal of R is a radical ideal. Hence, R is von Neumann regular. If $\Gamma(R) = \emptyset$, then R is a domain and therefore a field. If $\Gamma(R) \neq \emptyset$, then there is a nonzero idempotent zero-divisor e of R. Then, $R = eR \oplus (1-e)R$. Note that both eR and $(1-e)R$ are then isomorphic to fields.

(c) Suppose that R is a quasilocal ring with unique maximal ideal (x) such that $(x)^2 = (0)$. Then (x) is the only nonzero ideal of R, and therefore $\gamma(R) = \{\Gamma_{(x)}(R)\} = \{\emptyset\}$ by Proposition 2.

Conversely, suppose that $\gamma(R) = \{\emptyset\}$. This implies each nonzero ideal of R is a radical ideal. Since R is not reduced, $nil(R) \neq (0)$. The result now follows from Proposition 53. □

Orderings on the Vertices of $\Gamma_I(R)$

The following definitions are taken from [**14**].

DEFINITION 56. *Given a graph G with vertices a and b, we define the following relations.*

(a.) $a \leq b$ if every vertex adjacent to b is also adjacent to a.

(b.) $a \sim b$ if both $a \leq b$ and $b \leq a$, i.e., a and b are adjacent to the same set of vertices.

(c.) $a \perp b$ if a and b are adjacent and no vertex of G is adjacent to both a and b.

REMARK 57. *A vertex in a graph G is never considered to be self-adjacent. Thus, if $a \leq b$, then a is not adjacent to b (since otherwise b is self-adjacent).*

PROPOSITION 58. *Let I be an ideal of a ring R. Let $a, b \in R - I$ such that $b + I$ and $a + I$ represent nonconnected columns of $\Gamma_I(R)$. Then $a + I \leq b + I$ in $\Gamma(R/I)$ if and only if $a \leq b$ in $\Gamma_I(R)$.*

Proof : Suppose $a + I \leq b + I$ in $\Gamma(R/I)$. Let $x \in R - I$ be adjacent to b. Since $b + I$ is nonconnected, $x + I \neq b + I$ (otherwise, $(b+I)^2 = (x+I)(b+I) = xb+I = 0+I$). Thus, by Theorem 5, $x + I$ is adjacent to $b + I$. Therefore, $a + I \leq b + I$ implies $x + I$ is adjacent to $a + I$. Hence, again by Theorem 5, x is adjacent to a.

Conversely, suppose $a \leq b$ in $\Gamma_I(R)$. Let $x + I \in R/I$ be adjacent to $b+I$. Then, by Theorem 5, x is adjacent to b in $\Gamma_I(R)$. Since $a \leq b$, x is adjacent to a. Since $a + I$ is nonconnected, $x + I \neq a + I$. Thus, by Theorem 5, $x + I$ is adjacent to $a + I$. □

COROLLARY 59. *Let I be an ideal of a ring R. Let $a, b \in R - I$ such that $b + I$ and $a + I$ represent nonconnected columns of $\Gamma_I(R)$. Then $a + I \sim b + I$ in $\Gamma(R/I)$ if and only if $a \sim b$ in $\Gamma_I(R)$.*

COROLLARY 60. *Let I be an ideal of a ring R. Let $x, y \in a + I$, where $a + I$ represents a nonconnected column of $\Gamma_I(R)$. Then $x \sim y$ in $\Gamma_I(R)$.*

REMARK 61. *The conclusions of the above results fail if the columns are connected. The problem is that one could produce self-adjacent vertices, as mentioned in the previous remark.*

PROPOSITION 62. *Let I be an ideal of a ring R such that $\Gamma_I(R)$ is a graph on three or more vertices. Let $a, b \in R - I$ such that $a + I \neq b + I$ and $a + I, b + I$ both represent nonconnected columns of $\Gamma_I(R)$. Then $a \perp b$ in $\Gamma_I(R)$ if and only if $a + I \perp b + I$ in $\Gamma(R/I)$.*

Proof : Suppose $a \perp b$ in $\Gamma_I(R)$. Then a is adjacent to b, and so $a + I$ is adjacent to $b + I$ in $\Gamma(R/I)$ by Theorem 5. If $x \in R - I$ such that $x + I$ is adjacent to both $a + I$ and $b + I$, then, by Theorem 5, x is adjacent to both a and b. Hence, there can be no vertex $x + I$ of $\Gamma(R/I)$ adjacent to both $a + I$ and $b + I$.

Conversely, suppose $a + I \perp b + I$ in $\Gamma(R/I)$. Then a is adjacent to b in $\Gamma_I(R)$. Assume there is some $x \in R - I$ such that x is adjacent to both a and b. Then $xa, xb \in I$. Since $a + I, b + I$ are nonconnected, $a + I \neq x + I \neq b + I$. Therefore, we have $x + I$ is adjacent to both $a + I$ and $b + I$; contradicting $a + I \perp b + I$. Hence, no vertex x of $\Gamma_I(R)$ can be adjacent to both a and b. □

PROPOSITION 63. *Let I be a nonzero ideal of a ring R such that $\Gamma_I(R)$ is a graph on at least three vertices. If $c + I$ is a connected column of $\Gamma_I(R)$, then $c \not\perp y$ for any vertex $y \neq c$ of $\Gamma_I(R)$.*

Proof : Assume there exists $y \in R - I$ such that $y \neq c$ and $y \perp c$. By assumption, $yc \in I$.

<u>*Case 1 :*</u> $y + I \neq c + I$. Choose $0 \neq j \in I$. Then $y(c + j) \in I$ and $c(c + j) \in I$ since $c + I$ is a connected column. Thus $c + j$ is adjacent to both c and y.

Case 2 : $y + I = c + I$. Then $y = c + j$ for some $0 \neq j \in I$.

Subcase 1 : $|I| \geq 3$. Choose $0 \neq k \in I$ with $k \neq j$. Then, since $c + I$ is a connected column, $c + k$ is adjacent to both c and $y = c + j$.

Subcase 2 : $|I| = 2$. Since $\Gamma_I(R)$ is a graph on at least three vertices, $\Gamma_I(R)$ must consist of more than one column. Let $d \in R - I$ such that $d + I$ is adjacent to $c + I$ in $\Gamma(R/I)$. Then c and $y = c + j$ are both adjacent to d in $\Gamma_I(R)$ by Theorem 5.

Hence we get a contradiction in all cases. □

Note that if $\Gamma_I(R)$ is the connected graph on the two vertices x and y, then trivially $x \perp y$.

Bridges and the Degree of a Vertex

Let G be a connected graph. Then [7] defines an edge E of G to be a *bridge* if the graph $G - E$ is disconnected.

PROPOSITION 64. *Let I be an ideal of a ring R. Then $\Gamma_I(R)$ has a bridge if and only if either (a.) $\Gamma_I(R)$ is the graph on two vertices, or (b.) $I = (0)$ and $\Gamma(R)$ has a bridge.*

Proof : If either (a) or (b) hold, then $\Gamma_I(R)$ clearly has a bridge.

Suppose $\Gamma_I(R)$ has a bridge. If $I = (0)$, then $\Gamma_I(R) = \Gamma(R)$. So assume $I \neq (0)$.

Case 1 : $|I| = 2$ and $\Gamma(R/I)$ is a graph on one vertex. Then $\Gamma_I(R)$ is a graph on two vertices.

Case 2 : $|I| \geq 3$ and $\Gamma(R/I)$ is a graph on one vertex. Then $\Gamma_I(R)$ consists of a single connected column, and therefore $\Gamma_I(R)$ is a complete graph on $|I|$ vertices. But since $|I| \geq 3$, it is clear that $\Gamma_I(R)$ cannot have a bridge. This is a contradiction.

<u>*Case 3 :*</u> $|I| \geq 2$ and $\Gamma(R/I)$ is a graph on two or more vertices. Let a be a vertex of $\Gamma_I(R)$. For any vertex c adjacent to a, remove the edge E from a to c. To show $\Gamma_I(R) - E$ is connected, it suffices to find another path from a to c. If $a + I = c + I$, we can find $b \in R$ such that $b + I \neq a + I$ and $b + I$ is adjacent to $a + I$ in $\Gamma(R/I)$. Thus $a - b$ and $b - c$ are edges in $\Gamma_I(R) - E$. If $a + I \neq c + I$, for $0 \neq i \in I$ the edges $a - c + i$, $c + i - a + i$, and $a + i - c$ are all contained in $\Gamma_I(R) - E$. □

For a connected graph G we define

$$\delta(G) = min\{deg(x) |\ x\ is\ a\ vertex\ of\ G\}.$$

PROPOSITION 65. *Let I be an ideal of a commutative ring R.*
(a.) If $\Gamma(R/I)$ is a graph on one vertex, then $\delta(\Gamma_I(R)) = |I| - 1$.
(b.) If $\Gamma(R/I)$ is a graph on two or more vertices, then $\delta(\Gamma_I(R)) \geq \delta(\Gamma(R/I)) \cdot |I|$.

Proof : (a) If $\Gamma(R/I)$ has only one vertex, then $\Gamma_I(R)$ is a complete graph on $|I|$ vertices.

(b) If each vertex of $\Gamma_I(R)$ is adjacent to an infinite number of other vertices, the result is trivial. Also, if $|I| = \infty$, then the result is trivially true. So, suppose $|I| < \infty$. Let a be a vertex of $\Gamma_I(R)$ that is adjacent to only a finite number of other vertices. Let $\{b_1 + I, b_2 + I, \ldots, b_n + I\}$ be the set of all vertices of $\Gamma(R/I)$ that are adjacent to $a + I$. Then $n \geq \delta(\Gamma(R/I))$. Then the set of all vertices adjacent to a in $\Gamma_I(R)$ contains the set $\{b_j + i |\ i \in I,\ j = 1, \ldots, n\}$. Thus $deg(a) \geq \delta(\Gamma(R/I)) \cdot |I|$. (In fact, this inequality is strict if $a + I$ represents a connected column of $\Gamma_I(R)$ and equality holds if $a + I$ is not a connected column.) □

COROLLARY 66. *If* $a+I$ *is a nonconnected column of* $\Gamma_I(R)$ *and* $deg(a+I) = \delta(\Gamma(R/I))$ *in* $\Gamma(R/I)$*, then* $\delta(\Gamma_I(R)) = \delta(\Gamma(R/I)) \cdot |I|$.

CHAPTER 3

The Zero-Divisor Graph of a Commutative Ring

In this chapter, we give several results relating the ring-theoretic properties of a ring R to the structure of $\Gamma(R)$.

$\Gamma(R)$ for Rings Without Identity

In the original definition of the zero-divisor graph of a ring R, we only considered rings that have a multiplicative identity element, even though the definition did not require an identity. Given a ring R (with or without multiplicative identity), let $S = \mathbb{Z}$ if the characteristic of R is zero and let $S = \mathbb{Z}/n\mathbb{Z}$ if the characteristic of R is $n \neq 0$. Let $R_n = R \times S$, where n is the characteristic of R. Then R_n with multiplication defined by the rule $(a, m) \cdot (c, n) = (ac + mc + na, mn)$ contains R as a subring via the identification $r \mapsto (r, 0)$. Since R is a subring of R_n, $\Gamma(R)$ is an induced subgraph of $\Gamma(R_n)$.

THEOREM 1. *Let R be a ring of characteristic zero without identity. If $\Gamma(R)$ is not the empty graph, then $\Gamma(R) = \Gamma(R_0)$ if and only if for every $0 \neq a \in R$, $c \in R$ and $0 \neq m \in \mathbb{Z}$, we have $ac \neq ma$.*

Proof : Suppose that $\Gamma(R) = \Gamma(R_0)$. Let $0 \neq a \in R$, $c \in R$, and $0 \neq m \in \mathbb{Z}$. Then $(c, -m) \in R_0 - R$. Since $\Gamma(R) = \Gamma(R_0)$, $(c, -m)$ cannot be a vertex of $\Gamma(R_0)$. Therefore, $(c, -m)(a, 0) \neq (0, 0)$. Thus $ac - ma \neq 0$.

Suppose that for every $0 \neq a \in R$, $c \in R$ and $0 \neq m \in \mathbb{Z}$, we have $ac \neq ma$. Assume that (c, m) is a vertex of $\Gamma(R_0)$ for some $m \neq 0$. If (c, m) is a vertex, then there is some $(0,0) \neq (a, n) \in R_0$ such that $(a, n)(c, m) = (0, 0)$. Then, in particular, $mn = 0$, and therefore $n = 0$. But then we have $ac+ma = 0$, contradicting our hypothesis. Therefore, all vertices of $\Gamma(R_0)$ are contained in R, and hence $\Gamma(R) = \Gamma(R_0)$. $\square$

COROLLARY 2. *Let R be a ring of characteristic zero without identity. If $\Gamma(R) = \Gamma(R_0) \neq \emptyset$, then*

(a.) R has no nonzero idempotent elements.

(b.) R has no (additive) torsion zero-divisors.

(c.) $\Gamma(R)$ is a graph on an infinite number of vertices.

Proof : (a) and (b) are immediate from the above theorem.

(c) Let a be a vertex of $\Gamma(R)$. Then there is some $c \neq 0$ such that $ac = 0$. Then, for every $0 \neq m \in \mathbb{Z}$, $(ma)c = m(ac) = 0$. By (b), the collection $\{ma \mid 0 \neq m \in \mathbb{Z}\}$ is an infinite subset of the vertex set of $\Gamma(R)$. $\square$

By an analogous argument to that of Theorem 1, we can prove the following theorem.

THEOREM 3. *Let R be a ring without identity of characteristic p for some prime p. If $\Gamma(R)$ is not the empty graph, then $\Gamma(R) = \Gamma(R_p)$ if and only if for every $0 \neq a \in R$, $c \in R$, and $0 \neq m \in \mathbb{Z}_p$, we have $ac \neq ma$.*

REMARK 4. *Let R be a ring without identity of characteristic $n \neq 0$, where n is not prime. Then $\Gamma(R) \neq \Gamma(R_n)$.*

Proof : Choose nonzero elements m, k in $\mathbb{Z}_n$ such that $mk = 0$. Then $(0, m)$ and $(0, k)$ are vertices of $\Gamma(R_n)$ not contained in R since $(0, m)(0, k) = (0, 0)$. $\square$

EXAMPLE 5. 1. Let $R = \bigoplus_{i=1}^{\infty} \mathbb{Z}_{p_i}$, where p_i is the i^{th} prime. Then R is a ring of characteristic zero without a multiplicative identity and $\Gamma(R)$ is nonempty. Let $a = (1, 0, 0, \ldots)$ and $c = (0, 1, 0, \ldots)$. Then $a, c \in R$ and $ac = 0$. But $(c, 0), (2c, 1) \in R_0$ and $(c, 0)(2c, 1) = (2c^2 + c, 0) = (0, 0)$. Thus $(2c, 1)$ is a vertex of $\Gamma(R_0)$ and therefore, $\Gamma(R) \neq \Gamma(R_0)$. Note that $ac = 0 = 2a$, and therefore Theorem 1 also implies $\Gamma(R) \neq \Gamma(R_0)$.
2. Let $R = X\mathbb{Z}[X] \times X\mathbb{Z}[X]$. Then R is a ring of characteristic zero without multiplicative identity and $\Gamma(R)$ is nonempty. By Theorem 1, $\Gamma(R) = \Gamma(R_0)$. Let $R = X\mathbb{Z}_3[X] \times X\mathbb{Z}_3[X]$. By Theorem 3, $\emptyset \neq \Gamma(R) = \Gamma(R_3)$.

The Zero-Divisor Graph of a Module

Let R be a ring and M be an R-module. We form the idealization of R and M to create a ring A=R(+)M with operations $(r, m) + (s, n) = (r + s, m + n)$ and $(r, m) \cdot (s, n) = (rs, sm + rn)$ for all $r, s \in R$ and $m, n \in M$. We can view R as a subring of A via the embedding $r \mapsto (r, 0)$. Also, M can be identified with the ideal $0(+)M$ of A via the embedding $m \mapsto (0, m)$. Note that each $(0, m) \in A$ is nilpotent since $(0, m)^2 = (0, 0)$. If we define the zero-divisors of M to be the set $Z(M) = \{r \in R |\ rm = 0 \ for\ some\ m \in M^*\}$, then $Z(A) = (Z(R) \cup Z(M))(+)M$ (as seen in Theorem 25.3 of [**11**]).

DEFINITION 6. *Let R be a ring and let M be an R-module. Let A=$R(+)M$ as above. We define the zero-divisor graph of M, denoted by $\Gamma(M_R)$, to be $\Gamma(A)$.*

Note that if M is the zero module, then $A \simeq R$ and so $\Gamma(M_R) \simeq \Gamma(R)$. Therefore, all modules considered in this section are nonzero.

PROPOSITION 7. *Let R be a ring and M an R-module. Then $\Gamma(R)$ is an induced subgraph of $\Gamma(M_R)$, but $\Gamma(R) \neq \Gamma(M_R)$.*

Proof : Let a and b be adjacent vertices of $\Gamma(R)$. Then $(a,0), (b,0) \in A - \{0\}$ and $(a,0)(b,0) = (ab,0) = (0,0)$. Thus all vertices and edges of $\Gamma(R)$ are contained in $\Gamma(M_R)$. Since $(0,m)$ is a vertex of $\Gamma(M_R)$ for each $0 \neq m \in M$, $\Gamma(R) \neq \Gamma(M_R)$. Conversely, if a and b are adjacent vertices of $\Gamma(R)$, we have $(a,0)$ adjacent to $(b,0)$ in $\Gamma(M_R)$ since $(a,0)(b,0) = (ab,0) = (0,0)$. □

PROPOSITION 8. *Let R be a ring and M an R-module. Then $\omega(\Gamma(M_R)) \geq |M| - 1$.*

Proof : Let m and n be nonzero elements of M. Then $(0,m)(0,n) = (0,0)$. Thus the nonzero elements of M generate a complete subgraph of $\Gamma(M_R)$. □

COROLLARY 9. *Let R be an integral domain and let M be a torsion-free R-module. Then $\Gamma(M_R)$ is the complete graph on card$(M) - 1$ vertices.*

Consider $\mathbb{Z}_4$ as a $\mathbb{Z}$-module. The resulting graph, $\Gamma(M_R)$, is infinite since, for $n \neq 0$, $(2n, 0+4\mathbb{Z})$ and $(0, 2+4\mathbb{Z})$ are adjacent vertices. Figure 27 gives a finite example where $\Gamma(R) \not\cong \Gamma(M_R)$.

THEOREM 10. *Let R be a ring of connected type and let M be an R-module. Then A=$R(+)M$ is of connected type.*

Proof : Let J be a nontrivial ideal of A.

<u>*Case 1 :*</u> $0(+)M \not\subseteq J$. Then there is some $0 \neq m \in M$ such that $(0, m) \notin J$. But $(0, m)^2 = (0, 0)$. Therefore, $(0, m) + J$ represents a connected column of $\Gamma_J(A)$.

<u>*Case 2 :*</u> $0(+)M \subseteq J$. Let $X = \{a \in R |\ (a, m) \in J\ for\ some\ m \in M\}$. Note that $X \neq \emptyset$ since $0 \in X$.

<u>*Claim :*</u> X is an ideal of R.

Let $a, b \in X$. Then there exist $m, n \in M$ such that $(a, m), (b, n) \in J$. Then $a + b \in X$ since $(a + b, m + n) = (a, m) + (b, n) \in J$. For any $r \in R$, $ra \in X$ since $(ra, rm) = (r, 0)(a, m) \in J$.

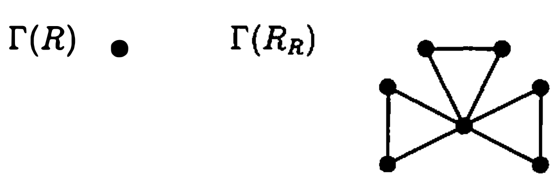

FIGURE 27. $\Gamma(R)$ and $\Gamma(R_R)$, where $R = \mathbb{Z}_4$.

Subcase 1 : $\Gamma_X(R) = \emptyset$. Then X is a prime ideal of R by Proposition 2 of Chapter 2. We show that $\Gamma_J(A) = \emptyset$. Suppose this is not the case; i.e., there is some $y \in R$ and $n \in M$ such that $(y, n) \in A - J$ and there is some $(x, m) \in A - J$ such that $(y, n)(x, m) \in J$. Thus $(xy, xn + ym) \in J$. So $xy \in X$. Now $\Gamma_X(R) = \emptyset$ implies either $x \in X$ or $y \in X$. If $x \in X$, then there is some $m_1 \in M$ such that $(x, m_1) \in J$. But then $(x, m) = (x, m_1) + (0, m - m_1) \in J$. If $y \in X$, we similarly obtain $(y, n) \in J$. Thus in either case we get a contradiction. Hence $\Gamma_J(A) = \emptyset$.

Subcase 2 : $\Gamma_X(R) \neq \emptyset$. Since R is of connected type, there is some $a \in R - X$ such that $(a+X)^2 = 0+X$; i.e., $a+X$ represents a connected column of $\Gamma_X(R)$. Since $a \notin X$, $(a, 0) \notin J$. But $a^2 \in X$ implies $(a^2, m) \in J$ for some $m \in M$. Thus $(a, 0)^2 = (a^2, m) - (0, m) \in J$. Therefore $(a, 0) + J$ is a connected column of $\Gamma_J(A)$.

All that remains to be shown is that $\gamma(A) \neq \{\emptyset\}$. Since R is of connected type, there is some nontrivial ideal I of R such that $\Gamma_I(R) \neq \emptyset$ and $\Gamma_I(R)$ has at least one connected column, say $a + I$. Let $J = I(+)M$. Then J is an ideal of A and $0(+)M \subseteq J$. Then, as in the above paragraph, $(a, 0) + J$ is a connected column of $\Gamma_J(A)$. In particular, $\Gamma_J(A) \neq \emptyset$. □

PROPOSITION 11. *If R has a nontrivial ideal I such that $IM \neq 0$ and $IM \neq M$, then A=R(+)M is not of nonconnected type.*

Proof : $L = 0(+)IM$ is an ideal of A. Let $m \in M - IM$. Since $(0, m)^2 = (0, 0)$, $(0, m)+L$ represents a connected column of $\Gamma_L(A)$. □

Nilpotent Elements

As seen in Chapter 2, the location of nilpotent elements in $\Gamma(R/I)$ has a significant effect on $\Gamma_I(R)$. In this section, we give some conditions that make the nilpotent elements of R somewhat easier to locate in $\Gamma(R)$.

Recall that the *degree* of a vertex x of a graph G, denoted $deg(x)$, is the number of vertices of G that are adjacent to x.

THEOREM 12. *Let R be a ring such that $\Gamma(R)$ is a graph on three or more vertices. If there is some $0 \neq x \in R$ such that $x^2 = 0$, then $deg(x) \geq 2$.*

Proof : Let $0 \neq y \in R$ be a vertex adjacent to x, i.e., $xy = 0$ and $y \neq x$. Note that $x(x + y) = 0$. Clearly, $x + y \neq x$ and $x + y \neq y$. If $y + x \neq 0$, we have $deg(x) \geq 2$. If $x + y = 0$, there must be some $z \in R - \{0, x, y\}$ such that x or y is adjacent to z since $\Gamma(R)$ has at least three vertices. Since $x = -y$, in either case we have $deg(x) \geq 2$. □

DEFINITION 13. *Let R be a ring. If $x \in R$ such that $x^n = 0$ and $x^j \neq 0$ for $1 \leq j \leq n-1$, we say x has nilpotency degree n. We denote this as $nildeg(x) = n$. If $y \in R$ is not nilpotent, let $nildeg(y) = 0$, and let $nildeg(0) = 1$.*

We can generalize the previous theorem as follows.

THEOREM 14. *Let R be a ring. If $x \in R$ and $nildeg(x) = n \geq 2$, then $deg(x^{n-1}) \geq 2^{n-1} - 2$.*

Proof : Let $A = \{b_1x + b_2x^2 + \ldots + b_{n-1}x^{n-1} | \ b_i = 0 \ or \ 1\}$. Then for all $a \in A$, $ax^{n-1} = 0$. If we show $|A| = 2^{n-1}$, then we are done since x^{n-1} is not adjacent to $0, x^{n-1} \in A$.

Suppose $x^{a_1} + x^{a_2} + \ldots + x^{a_k} = x^{c_1} + x^{c_2} + \ldots + x^{c_m}$, where $1 \leq a_1 < \ldots < a_k \leq n-1$ and $1 \leq c_1 < \ldots < c_m \leq n-1$. Without loss of generality, $a_1 \leq c_1$. Let $M = min\{a_2, \ldots, a_k, c_1, \ldots, c_m\}$. Note that $M \geq a_1$. Then $x^{n-M}x^{a_1} = x^{n-M}(x^{c_1} + \ldots + x^{c_m} - x^{a_2} - \ldots - x^{a_k})$. Each term on the right-hand side has degree at least n, and therefore is zero. Thus, the left-hand side must also have exponent at least n. Hence, $M \leq a_1$. So we have $a_1 = M = c_1$. By induction, we may now show that $k = m$ and $c_j = a_j$ for $j = 2, \ldots, k$. □

Figure 28 shows that equality may hold in Theorem 14.

COROLLARY 15. *If there is some integer $k \geq 2$ such that $deg(y) < 2^k - 2$ for every vertex y of $\Gamma(R)$, then $nildeg(x) \leq k$ for all $x \in R$.*

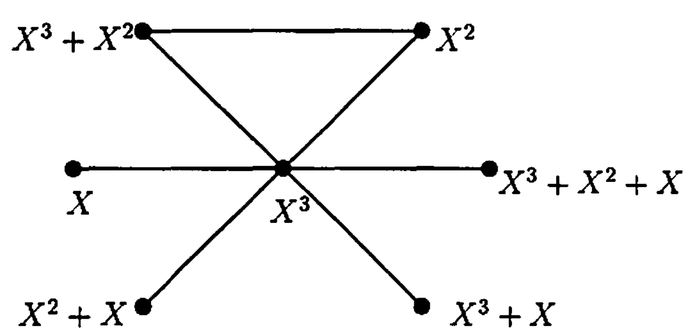

FIGURE 28. $\Gamma(R)$, where $R = \mathbb{Z}_2[X]/(X^4)$. Note that $nildeg(X^2) = 2$ and $deg(X^2) = 2$. Also, $nildeg(X) = 4$ and $deg(X^3) = 6$. Both these facts show that equality can hold in Theorem 14.

COROLLARY 16. *If $x \in R$ with $nildeg(x) = n \geq 2$, then x is adjacent to a vertex in $\Gamma(R)$ of degree at least $2^{n-1} - 2$.*

We can refine the above theorem for nilpotency degree three.

THEOREM 17. *Let R be a ring such that $\Gamma(R)$ is a graph on four or more vertices. If there is some $x \in R$ with $nildeg(x) = 3$, then $deg(x^2) \geq 3$.*

Proof : $(x^2)^2 = 0$ implies, by Theorem 12, $deg(x^2) \geq 2$. Clearly, x^2 is adjacent to x. So, there is a vertex $y \neq x$ or x^2 such that x^2 is adjacent to y.

Assume $deg(x^2) = 2$ and we find a contradiction. Since $\Gamma(R)$ has at least four vertices, there is some vertex w of $\Gamma(R)$ not equal to x, x^2, or y such that w is adjacent to x or y. Note that $wx \neq 0$; since otherwise $wx^2 = 0$ and therefore $deg(x^2) > 2$. Thus $wy = 0$. Now $(y+x^2)x^2 = 0$. Thus $y + x^2 \in \{0, x, x^2, y\}$. Clearly, $y + x^2 \neq y$ and $y + x^2 \neq x^2$. Also, $y + x^2 \neq 0$; since otherwise $wx^2 = w(-y) = -wy = 0$ and therefore $deg(x^2) > 2$. Thus, $y + x^2 = x$. Then $wx - wx^2 = w(x - x^2) = wy = 0$, and so $wx = wx^2$. Then $wx^2 = (wx)x = (wx^2)x = wx^3 = 0$, which implies $deg(x^2) > 2$. This is a contradiction. □

Figure 29 shows that we may have strict inequality in Theorems 14 and 17.

If $\Gamma(R)$ is bipartite, we can easily determine the nontrivial nilpotent elements of R.

THEOREM 18. *Let R be a ring. If $\Gamma(R) \simeq K^{n,m}$ for $n \geq 2$ and $m \geq 2$, then R contains no nonzero nilpotent elements.*

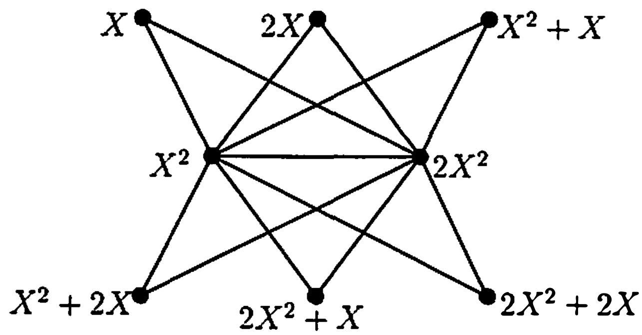

FIGURE 29. $\Gamma(R)$, where $R = \mathbb{Z}_3[X]/(X^3)$. Note that $nildeg(X) = 3$ and $deg(X^2) = 7$, showing strict inequality in Theorems 14 and 17.

Proof : Since $\Gamma(R)$ is bipartite, $Z(R) = \{0, a_1, \ldots, a_n, b_1, \ldots, b_m\}$, where $a_i b_j = 0$ for all i and j and $a_i a_j \neq 0$, $b_i b_j \neq 0$ whenever $i \neq j$.

Assume that R has a nonzero nilpotent element, and we find a contradiction. By assumption, R has some nilpotent element of nilpotency degree two. Say, without loss of generality, $a_1^2 = 0$. Then $a_1(a_1 + b_j) = 0$ for $1 \leq j \leq m$. Thus, $a_1 + b_j \in Z(R)$ for each j. Choose j such that $a_1 + b_j \neq 0$. If $a_1 + b_j = a_k$ for some $1 \leq k \leq n$, then $a_k a_1 = (a_1 + b_j)a_1 = 0$. If $a_1 + b_j = b_i$ for some $1 \leq i \leq m$, then $a_n a_1 = a_n(b_i - b_j) = a_n b_i - a_n b_j = 0$. In either case, we get a contradiction. □

Using the same argument, we can prove the following theorem.

THEOREM 19. *Let R be a ring. If $\Gamma(R)$ is an infinite complete bipartite graph but not a star graph, then R has no nonzero nilpotent elements.*

The next results are a restatement of Lemma 39 and Lemma 43 of Chapter 2.

PROPOSITION 20. *Let R be a ring. If $\Gamma(R) \simeq K^{1,1}$, i.e., $\Gamma(R)$ is the graph on two vertices, then R has either no or exactly two nonzero nilpotent elements.*

PROPOSITION 21. *Let R be a ring. If $\Gamma(R) \simeq K^{1,2}$, then R has either no or exactly one nonzero nilpotent elements.*

THEOREM 22. *Let R be a ring. If $\Gamma(R) \simeq K^{1,r}$ for some integer $r \geq 3$, then R has no nonzero nilpotent elements.*

Proof : Since $r \geq 3$, $\Gamma(R)$ is a graph on at least four vertices. Then $R \simeq \mathbb{Z}_2 \times F$, where F is a finite field by Theorem 2.13 of [**2**]. □

In Theorem 1.14 of [**8**], it is shown that if R is a ring such that $\Gamma(R)$ is complete bipartite, then either $R \simeq K \oplus L$ for finite fields K and L (if R is finite), or R is a subdirect sum of $S \oplus T$ where S and T are integral domains with more than one element (if R is infinite). Therefore, the above results could also have been proven as corollaries to this theorem. A similar result concerning which rings yield complete bipartite graphs can be found in [**3**].

Orderings on the Vertices of a Zero-Divisor Graph

We recall the following definitions from Chapter 2.

DEFINITION 23. *Given a graph G with vertices a and b, we define the following relations.*

(a.) $a \leq b$ *if every vertex adjacent to* b *is also adjacent to* a.

(b.) $a \sim b$ *if both* $a \leq b$ *and* $b \leq a$, *i.e.,* a *and* b *are adjacent to the same set of vertices.*

(c.) $a \perp b$ *if* a *and* b *are adjacent and no vertex of* G *is adjacent to both* a *and* b.

In Remark 57 of Chapter 2, we noted that if $a \leq b$, then a and b cannot be adjacent vertices in G.

PROPOSITION 24. *Let* R *be a ring and let* $x \in R$ *with* $nildeg(x) = n \geq 3$. *If* $0 \neq b \in R$ *with* $b \neq x$ *or* x^{n-1}, *then* $b \not\perp x$.

Proof : If b is adjacent to x, then $xb = 0$. Therefore, x^{n-1} is adjacent to both b and x. □

PROPOSITION 25. *Let* R *be a ring and let* $x \in R$ *with* $nildeg(x) = n \geq 3$. *If* $x^{n-1} \neq -x^{n-1}$, *then* $x \not\perp b$ *for any vertex* b *in* $\Gamma(R)$.

Proof : If b is adjacent to x, then $xb = 0$. Let $y = x^{n-1}$ if $b \neq x^{n-1}$ and let $y = -x^{n-1}$ otherwise. Then y is adjacent to both b and x. □

In Figure 28, we have a ring of characteristic 2 where $nildeg(X) = 4$ and $X \perp X^3$.

PROPOSITION 26. *Let* R *be a ring and let* $y \in R$ *with* $nildeg(y) = 2$. *If* $y \neq -y$, *then for any vertex* $b \neq -y$ *of* $\Gamma(R)$, $y \not\perp b$.

Proof : If b is adjacent to y, then $yb = 0$. Then $-y$ is adjacent to both y and b. □

COROLLARY 27. *Let* R *be a ring and let* $y \in R$ *with* $nildeg(y) = 2$. *If* $\Gamma(R)$ *is a graph on at least three vertices and* $y \neq -y$, *then* $-y \not\perp y$.

Proof : By Theorem 12, $deg(y) \geq 3$. Thus there is some $a \in R - \{0, y, -y\}$ such that a is adjacent to y, and therefore also adjacent to $-y$. □

$\Gamma(\mathbb{Z}_9)$ is a graph on the two vertices $3 + 9\mathbb{Z}$ and $6 + 9\mathbb{Z} = -3 + 9\mathbb{Z}$ where $nildeg(3 + 9\mathbb{Z}) = 2$ and $3 + 9\mathbb{Z} \perp -3 + 9\mathbb{Z}$. Thus the conclusion of Corollary 27 may fail if $\Gamma(R)$ does not have at least three vertices.

Figures 30 and 31 show that a result analogous to Proposition 26 cannot be found for rings of characteristic 2.

Given a graph G, it is trivial to verify that $\sim$ is an equivalence relation on the set of vertices of G. For a vertex a of G, we define the $\sim$ *equivalence class of a*, denoted $\overline{a}$, to be the set of all vertices b of G such that $a \sim b$. Note that $a \in \overline{a}$.

DEFINITION 28. *Let G be a graph. We define the $\sim$ − equivalence class graph of G, denoted ecg(G), to be the graph with vertex set* $\{\overline{b} \mid b \text{ is a vertex of } G\}$, *where $\overline{a}$ is adjacent to $\overline{b}$ in ecg(G) if and only if a is adjacent to b in G. (It is trivial to verify that this graph is well-defined.)*

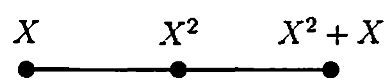

FIGURE 30. $\Gamma(R)$, where $R = \mathbb{Z}_2[X]/(X^3)$. Then $char(R) = 2$, $nildeg(X^2) = 2$, and $X^2 \perp X$, and $X^2 \perp X^2 + X$.

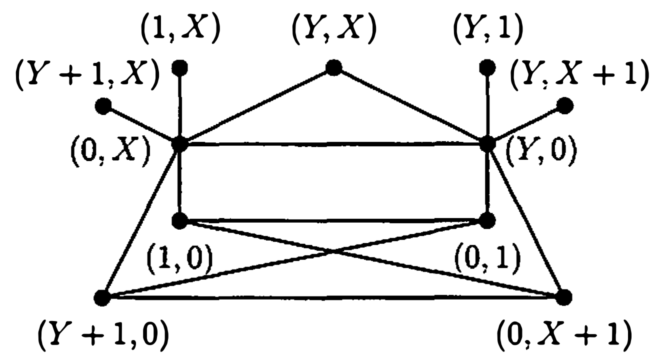

FIGURE 31. $\Gamma(R)$, where $R = \mathbb{Z}_2[Y]/(Y^2)\times\mathbb{Z}_2[X]/(X^2)$. Then $char(R) = 2$, $nildeg(Y, X) = 2$, and $(Y, X) \not\leq a$ for any $a \in R - \{(Y, X)\}$.

Note that $ecg(G)$ is isomorphic to a subgraph of G. If G is a complete graph, then $G \simeq ecg(G)$. If G is a complete bipartite graph, then $ecg(G)$ is the connected graph on two vertices.

PROPOSITION 29. *Let G be a connected graph. Then $ecg(G)$ is connected and $diam(ecg(G)) \leq diam(G)$.*

Proof : Let $\overline{a}$ and $\overline{b}$ be two vertices of $ecg(G)$. Since G is connected, there is a path $a - x_1 - \ldots - x_n - b$ in G. Then $\overline{a} - \overline{x_1} - \ldots - \overline{x_n} - \overline{b}$ is a path in $ecg(G)$. Note that here we may have a path that contains a cycle since it could be the case that $\overline{x_i} = \overline{x_j}$. This only means that $d(\overline{a}, \overline{b}) \leq d(a, b)$. Hence, $diam(ecg(G)) \leq diam(G)$. □

COROLLARY 30. *Let R be a ring. Then $ecg(\Gamma(R))$ is connected and $diam(ecg(\Gamma(R))) \leq 3$.*

For a ring R, let $N(R)$ denote the set of non-zero-divisors of R. That is, $N(R) = \{x \in R |\ if\ c \in R\ with\ xc = 0,\ then\ c = 0\}$. Some authors refer to $N(R)$ as the set of *regular elements* of R. Note that $\{1\} \subseteq U(R) \subseteq N(R)$.

PROPOSITION 31. *Let R be a ring. If $a \in Z(R)^*$ with $nildeg(a) \neq 2$, then $xa \sim a$ for all $x \in N(R)$.*

Proof : First note that, for any $x \in N(R)$, $a(xa) \neq 0$ since $a^2 \neq 0$. Thus a is not adjacent to xa.

Let $x \in N(R)$. If b is adjacent to a, $ab = 0$. Thus $xab = 0$.

If c is adjacent to xa, then $x(ac) = (xa)c = 0$. Since $x \in N(R)$, we must have $ac = 0$. □

PROPOSITION 32. *Let R be a ring. If $a \in Z(R)^*$ with $nildeg(a) = 2$ and $a \neq -a$, then $|\overline{a}| = 1$.*

Proof : Assume $b \sim a$ for some $b \in R - \{a\}$. Note that $(-a)a = -(a^2) = 0$ implies a is adjacent to $-a$. Thus b is adjacent to $-a$. That is, $-(ab) = (-a)b = 0$. But then b is adjacent to a, contradicting $b \sim a$. □

Figures 32, 33, 34, and 35 give examples of $\Gamma(R)$ and $ecg(\Gamma(R))$ for various rings R. Several more examples are given in Figures 41 through 54 of the Appendix.

A different notion of equivalent vertices was used by A. Lauve in [**13**]. In his work, vertices x and y of $\Gamma(R)$ were equivalent if and only if $x = uy$ for some $u \in U(R)$. This definition of the equivalence of x

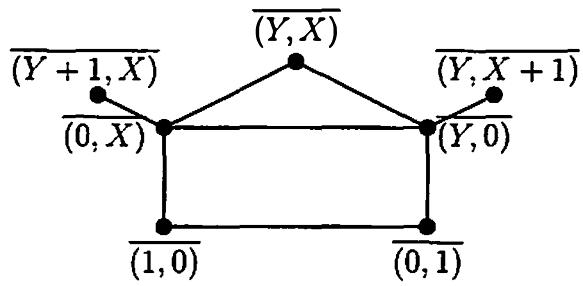

FIGURE 32. $ecg(\Gamma(R))$, where $R = \mathbb{Z}_2[Y]/(Y^2) \times \mathbb{Z}_2[X]/(X^2)$. Compare this to $\Gamma(R)$ in Figure 31.

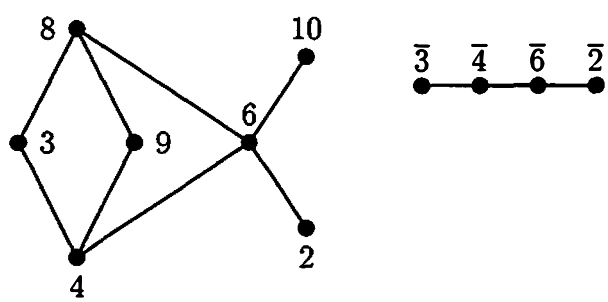

FIGURE 33. $\Gamma(\mathbb{Z}_{12})$ and $ecg(\Gamma(\mathbb{Z}_{12}))$.

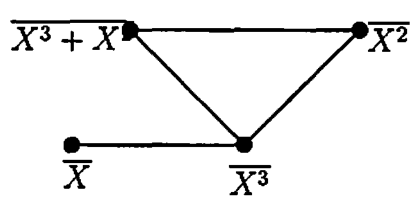

FIGURE 34. $ecg(\Gamma(R))$, where $R = \mathbb{Z}_2[X]/(X^4)$. $\Gamma(R)$ was given in Figure 28.

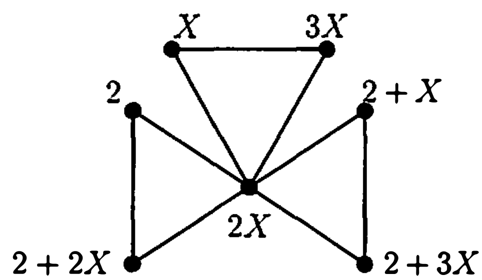

FIGURE 35. Let $R = \mathbb{Z}_4[X]/(X^2)$. Then $\Gamma(R) \simeq ecg(\Gamma(R))$.

and y implies $x \sim y$ if $nildeg(x) \neq 2$, but Figure 36 shows the converse fails.

REMARK 33. *Note that for a ring R and distinct $x, y \in R$, $x \sim y$ in $\Gamma(R)$ is equivalent to (1.) $xy \neq 0$, (2.) for all $a \in R - \{x\}$ such that $ax = 0$, we also have $ay = 0$, and (3.) for all $b \in R - \{y\}$ such that $by = 0$, we also have $bx = 0$. That is, for any $x, y \in Z(R)^*$, $x \sim y$ if and only if $ann(x) - \{x\} = ann(y) - \{y\}$.*

In light of the above remark, a few results about annihilator ideals are useful here.

PROPOSITION 34. *Let R be a ring and let $x, y \in R^*$ such that $ann(y) = ann(x)$. Then $ann(x^k) = ann(y^k)$ and $ann(x^j y^k) = ann(x^{k+j}) = ann(y^{k+j})$, for any $k, j \geq 1$.*

Proof : Let $b \in ann(x^k)$. Then $bx^{k-1} \in ann(x) = ann(y)$, and hence $bx^{k-1}y = 0$. Thus $bx^{k-2}y \in ann(x) = ann(y)$. So $bx^{k-2}y^2 = 0$.

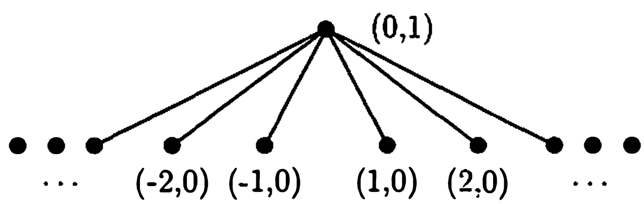

FIGURE 36. $\Gamma(\mathbb{Z} \times \mathbb{Z}_2)$. Note that $(1,0) \sim (n,0)$ for any integer $n \neq 0$.

By continuing in this fashion, we obtain $by^k = 0$, and thus $b \in ann(y^k)$. Hence, $ann(x^k) \subseteq ann(y^k)$. Similarly, one can show $ann(y^k) \subseteq ann(x^k)$.

Let $b \in ann(x^j y^k)$. Then $bx^j y^{k-1} \in ann(y) = ann(x)$. Hence, $bx^{j+1}y^{k-1} = 0$. Now, by continuing in the same manner as in the above paragraph, we obtain $bx^{k+j} = 0$. Thus $ann(x^j y^k) \subseteq ann(x^{k+j})$.

Let $a \in ann(x^{k+j})$. Then $ax^{k+j-1} \in ann(x) = ann(y)$. So $ax^{k+j-1}y = 0$. By continuing as in the previous cases, we have $ax^k y^j = 0$. Thus $ann(x^{k+j}) \subseteq ann(x^j y^k)$. Therefore, $ann(x^{k+j}) = ann(x^j y^k) = ann(y^{k+j})$.

□

COROLLARY 35. *Let R be a ring. If $x \sim y$ in $\Gamma(R)$ for non-nilpotent $x, y \in R$, then $ann(x) = ann(y)$ and $x^k \sim y^k$ for every integer $k \geq 1$.*

Proof : Since $x \sim y$, we have $ann(x) - \{x\} = ann(y) - \{y\}$. If x and y are non-nilpotent, then $x \notin ann(x)$ and $y \notin ann(y)$. Thus, $ann(x) = ann(y)$. The result now follows from the above proposition since $x^k \notin ann(x^k)$ and $y^k \notin ann(y^k)$. □

COROLLARY 36. *Let R be a ring. If $x \sim y$ in $\Gamma(R)$ for distinct nilpotent $x, y \in R$, then $ann(x) = ann(y)$ and $nildeg(x) = nildeg(y) > 2$.*

Proof : Since $x \sim y$, $xy \neq 0$ and $ann(x) - \{x\} = ann(y) - \{y\}$. Assume $x^2 = 0$. Since $\Gamma(R)$ is connected, there must be some $a \in Z(R)^*$ such that a is adjacent to both x and y (if not, $\Gamma(R)$ consists of the two isolated vertices x and y). Now $x(x+a) = x^2 + xa = 0 + 0 = 0$ and $x + a \neq x$. Thus, $x + a \in ann(x) - \{x\} = ann(y) - \{y\}$. So $0 = y(x + a) = yx + ya = yx + 0$. This contradicts $xy \neq 0$. Hence, $nildeg(x) > 2$, and so $x \notin ann(x)$. Similarly, $nildeg(y) > 2$, and $y \notin ann(y)$. Hence, $ann(x) = ann(y)$.

Let $n = nildeg(x)$ and $m = nildeg(y)$. Then $ann(y^n) = ann(x^n) = ann(0) = R$. But $ann(y^n) = R$ implies $y^n = 0$. Thus $m \leq n$. By a similar argument, $n \leq m$. □

COROLLARY 37. *Let R be a ring. If $0 \neq x \in R$ is nilpotent and $0 \neq y \in R$ is not nilpotent, then $x \not\sim y$.*

Proof : Assume $x \sim y$. Then $xy \neq 0$ and $ann(x) - \{x\} = ann(y) - \{y\}$. Since y is not nilpotent, $y \notin ann(y)$. If $x^2 = 0$, we get the same contradiction arrived at in the previous proof. Therefore, $x \notin ann(x)$. Hence, $ann(x) = ann(y)$. Let $n = nildeg(x)$. Then $ann(y^n) = ann(x^n) = ann(0) = R$. But this implies $y^n = 0$, a contradiction since y is not nilpotent. □

We may now strengthen Remark 33 as follows.

THEOREM 38. *Let R be a ring. Then $x \sim y$ in $\Gamma(R)$ if and only if either $xy \neq 0$ and $ann(x) = ann(y)$ or $x = y$.*

Proof : Suppose $x \sim y$ for distinct x and y. Then $xy \neq 0$. By Corollary 37, either both x and y are non-nilpotent or both x and y are nilpotent. Corollaries 35 and 36 show that $ann(x) = ann(y)$ in each case.

Conversely, suppose $xy \neq 0$ and $ann(x) = ann(y)$ for distinct $x, y \in Z(R)^*$. Let a be a vertex of $\Gamma(R)$ adjacent to x (some such vertex must exist since $\Gamma(R)$ is connected). Thus, $ax = 0$ and so, by hypothesis, $a \neq y$. Hence, $a \in ann(x) = ann(y)$. Thus $ay = 0$, and therefore a is adjacent to y. Similarly, one may show any vertex b adjacent to y must also be adjacent to x. Thus $x \sim y$. $\square$

One might wish to extend our definition of $a \sim b$ to include the case where a could be adjacent to b. A first effort might be to define a relation on purely graphical terms: given a graph G with vertices x and y, $x \approx y$ if and only if either (1.) $x \sim y$ or (2.) x is adjacent to y in G and $x \sim y$ in the subgraph G^* obtained from G by removing the edge $x - y$. This relation is easily verified to be an equivalence relation. Since we are dealing with zero-divisor graphs throughout the rest of this thesis, we restate our definition of $\approx$ in terms of ring-theoretic properties.

DEFINITION 39. *Let R be a ring. We define a relation $\approx$ on the vertices of $\Gamma(R)$ by saying $x \approx y$ if and only if (1.) for all $a \in R - \{x, y\}$ such that $ax = 0$, we also have $ay = 0$, and (2.) for all $b \in R - \{x, y\}$*

such that $by = 0$, we also have $bx = 0$. That is, $x \approx y$ if and only if $ann(x) - \{x, y\} = ann(y) - \{x, y\}$.

For a ring R and vertex a of $\Gamma(R)$, we define the <u>$\approx$ *equivalence class of* a</u>, denoted $\bar{\bar{a}}$, to be the set of all vertices b of $\Gamma(R)$ such that $a \approx b$. Note that $a \in \bar{\bar{a}}$.

DEFINITION 40. *Let G be a graph. We define the* <u>$\approx$ *− equivalence class graph of* G</u>*, denoted $ECG(G)$, to be the graph with vertex set $\{\bar{\bar{b}} \mid b$ is a vertex of $G\}$, where $\bar{\bar{a}}$ is adjacent to $\bar{\bar{b}}$ in $ECG(G)$ if and only if a is adjacent to b in G. (It is trivial to verify that this graph is well-defined.)*

Note that $ECG(G)$ is isomorphic to a subgraph of G. If G is a complete bipartite graph, then $ECG(G)$ is the connected graph on two vertices. This gives an example where $ECG(G) \simeq ecg(G)$. Note that $ECG(G)$ is always isomorphic to a subgraph of $ecg(G)$. Figure 37, Figure 38, and the next proposition each give an example in which $ECG(G) \not\simeq ecg(G)$.

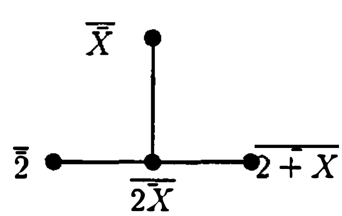

FIGURE 37. Let $R = \mathbb{Z}_4[X]/(X^2)$. Then $\Gamma(R) \not\simeq ECG(\Gamma(R))$. In Figure 35, we showed $\Gamma(R) \simeq ecg(\Gamma(R))$.

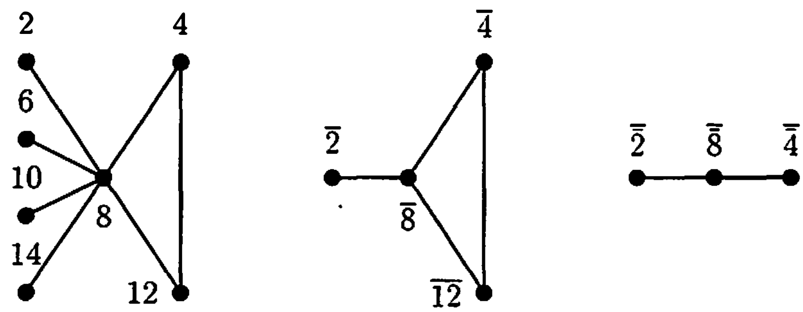

FIGURE 38. Let $R = \mathbb{Z}_{16}$. Below are $\Gamma(R)$, $ecg(\Gamma(R))$, and $ECG(\Gamma(R))$, respectively.

PROPOSITION 41. *Let G be a connected graph. Then G is complete if and only if $ECG(G)$ consists of a single vertex.*

Proof : If G is complete, it is clear that $a \approx b$ for every pair of vertices a and b from G.

Suppose that $ECG(G)$ consists of a single vertex, i.e., $a \approx b$ for every pair of vertices a and b in G. Assume that G has vertices a and b which are not adjacent. Since $a \approx b$, we must have $a \sim b$. Therefore, there must be some vertex c of G distinct from a and b such that a is adjacent to c and b is adjacent to c (if no such c exists, then G is a disconnected graph). By hypothesis, $a \approx c$. Since c is adjacent to a, it must be the case that $a \sim c$ in the subgraph G^* obtained from G by removing the edge $a - c$. Note that c is adjacent to b in G^*. Thus, a is adjacent to b in G^*. But this contradicts the fact that a and b are not adjacent in G. Hence, G must be complete. $\square$

THEOREM 42. *Let R be a ring. Let $x, y \in Z(R)^*$ such that $x \not\sim y$ and $x \approx y$ in $\Gamma(R)$. Then $xy = 0$ and either (a.) $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, or (b.) $nildeg(x) = nildeg(y) = 2$ and $ann(x) = ann(y)$.*

Proof : Note that $x \not\sim y$ implies $x \neq y$. It is clear from the definition of $\sim$ and $\approx$ that $xy = 0$ and $ann(x) - \{x, y\} = ann(y) - \{x, y\}$.

<u>Case 1:</u> Suppose neither x nor y is nilpotent. Then $x^2 \in ann(y)$, but $x^2 \notin ann(x)$. Thus, by Definition 39, either $x^2 = x$ or $x^2 = y$. But $x^2 \neq y$ (otherwise, $y^2 = yx^2 = 0$). Therefore, $x^2 = x$, i.e., x is an idempotent. Since x is a zero-divisor, $x \notin U(R)$. Then $1 - x$ is nonzero and a non-unit since $x(1 - x) = 0$. Thus, $1 - x \in ann(x)$. But $y(1 - x) = y - xy = y \neq 0$. So $1 - x \notin ann(y)$. Thus either $1 - x = x$ or $1 - x = y$. But $1 - x \neq x$ (otherwise, $1 = 2x$ and so $x \in U(R)$). Hence $1 - x = y$.

Now, let $a \in ann(x) - \{x, y\}$. Then $0 = ax + ay = a(x + y) = a(x + 1 - x) = a$. Hence, $ann(x) = \{0, y\}$ and $ann(y) = \{0, x\}$. Thus, $\Gamma(R)$ must be the connected graph on the two vertices x and y (if not, there is some other vertex $z \neq 0$ such that z is adjacent to either x or y, implying $z \in ann(x)$ or $z \in ann(y)$). By Example 2.1 (a) of [**3**], up to isomorphism, the only rings whose zero-divisor graph is the connected graph on two vertices are $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^3)$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since the ring R in question cannot have any nonzero nilpotent elements, we must have $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

<u>Case 2:</u> Suppose x is nilpotent. First, we show y is also nilpotent. Assume that y is not nilpotent. Note that $0 \neq y^2 \in ann(x)$, but

$y^2 \notin ann(y)$. Therefore, by Definition 39, either $y^2 = x$ or $y^2 = y$. But $y^2 \neq x$ since x is nilpotent and y is assumed not to be nilpotent. Therefore, $y^2 = y$, i.e., y is an idempotent. Then, as in Case 1, $1 - y \in ann(y)$ and $x(1-y) = x - xy = x \neq 0$. Thus $1-y \notin ann(x)$. Therefore, $1 - y = y$ or $1 - y = x$. But $1 - y \neq y$ since, as in Case 1, $y \notin U(R)$. Also, $1 - y \neq x$ since x is nilpotent and $1 - y$ is idempotent. This is a contradiction. Hence, y must be nilpotent.

Let $nildeg(x) = n$ and $nildeg(y) = m$. We show that $n = m = 2$. Assume $n > 2$ and $m > 2$. By definition of n, $x^{n-2} \in ann(y)$, but $x^{n-2} \notin ann(x)$. Therefore, by Definition 39, either $x^{n-2} = y$ or $x^{n-2} = x$. But $x^{n-2} \neq y$ (otherwise, $y^2 = yx^{n-2} = 0$). Thus $x^{n-2} = x$. Then $x^3 = x^n = 0$ (i.e., $n = 3$). Note that $x^2 \neq x$ (otherwise, $x^2 = (x^2)^2 = x^3x = 0$). Also note that $x - x^2 \in ann(y)$, but $x - x^2 \notin ann(x)$. Therefore, either $x - x^2 = x$ or $x - x^2 = y$. But clearly, $x - x^2 \neq x$. So $x - x^2 = y$. Now $y^3 = (x - x^2)^3 = 0$, i.e., $m = 3$. Thus $y(x + y^2) = 0$, but $x(x + y^2) = x^2 \neq 0$. So $x + y^2 \in ann(y)$ and $x + y^2 \notin ann(x)$. Therefore, either $x + y^2 = x$ or $x + y^2 = y$. Clearly, $x + y^2 \neq x$. Also, $x + y^2 \neq y$ (otherwise, $y^2 = y(x + y^2) = 0$). Thus we have a contradiction.

So, without loss of generality, $m = 2$. Then $y(x + y) = xy + y^2 = 0$ and $x(x + y) = x^2 + xy = x^2$. Thus, $x + y \in ann(y)$. Assume $n > 2$. Then $x + y \notin ann(x)$. Therefore, by Definition 39, either $x + y = x$ or $x + y = y$. This is a contradiction. Thus $n = 2$.

Note that in this case $x, y \in ann(x)$ and $x, y \in ann(y)$. Thus $ann(x) = ann(y)$. □

We can strengthen the condition in Definition 39 for $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

COROLLARY 43. *Let R be a ring such that $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, for vertices x and y of $\Gamma(R)$, $x \approx y$ if and only if $ann(x) = ann(y)$.*

Proof : Suppose $x \approx y$. If $x \sim y$, then $ann(x) = ann(y)$ by Theorem 38. If $x \approx y$ and $x \not\sim y$, then $ann(x) = ann(y)$ by the previous theorem.

Conversely, suppose $ann(x) = ann(y)$. Then $ann(x) - \{x, y\} = ann(y) - \{x, y\}$. Hence, $x \approx y$. □

It is now clear that, for a ring $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the relation $\approx$ agrees with the equivalence relation used in 3.5 of [**16**].

COROLLARY 44. *Let $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring such that $ECG(\Gamma(R)) \neq ecg(\Gamma(R))$. Then R contains at least two distinct nonzero nilpotent elements.*

Proof : Since $ECG(\Gamma(R))$ can be naturally embedded into $ecg(\Gamma(R))$, we must have some vertex $\overline{a}$ contained in $ecg(\Gamma(R))$, but not in $ECG(\Gamma(R))$. Thus, there is some $b \in R$ such that $a \in \overline{\overline{b}}$, but $a \notin \overline{b}$. Clearly, $a \neq b$. Now apply the above theorem. □

COROLLARY 45. *Let $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring with at most one nonzero nilpotent element. Then, for any $a, b \in Z(R)^*$, $a \sim b$ if and only if $a \approx b$. Therefore, $ECG(\Gamma(R)) = ecg(\Gamma(R))$.*

Figure 39 shows that the converse of the Corollary 45 may fail.

COROLLARY 46. *Let I be a nonzero proper ideal of R. If $Rad(I) = I$ and $R/I \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $ECG(\Gamma(R/I)) = ecg(\Gamma(R/I))$.*

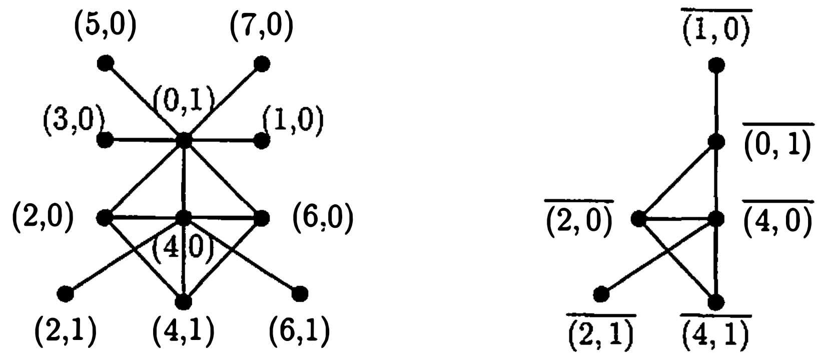

FIGURE 39. Let $R = \mathbb{Z}_8 \times \mathbb{Z}_2$. Then $ECG(\Gamma(R)) = ecg(\Gamma(R))$. Since $(4,0), (2,0), (6,0) \in R$ are nilpotent, this shows the converse of Corollary 45 can fail.

COROLLARY 47. *Let R be a ring. If there is some nilpotent $0 \neq x \in R$ with $nildeg(x) = n$ and $2x^{n-1} \neq 0$, then $ECG(\Gamma(R)) \neq ecg(\Gamma(R))$.*

Proof : Note that $x^{n-1} \approx -x^{n-1}$, but $x^{n-1} \not\sim -x^{n-1}$. □

REMARK 48. *We give an example of a ring R where $char(R) = 2$, R has an infinite number of nilpotent elements, and $ECG(\Gamma(R)) = ecg(\Gamma(R))$. Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_2[X_i]/(X_i^2)$. Assume $ECG(\Gamma(R))$ is not equal to $ecg(\Gamma(R))$. Then there exist $a, b \in R$ such that $a \approx b$ and $a \not\sim b$. Therefore, $ab = 0$ and $a^2 = 0 = b^2$ by Theorem 42. Thus the i^{th} coordinate of a and b must be either X_i or 0. Since $a \not\sim b$ implies $a \neq b$, there is some coordinate j such that $a_j \neq b_j$; say without loss of generality, $a_j = X_j$ and $b_j = 0$. Let $c = \{\delta_{i,j}\}_{i=1}^{\infty}$. Then c is a vertex of $\Gamma(R)$ such that c is adjacent to b, but not a. This contradicts $a \approx b$. Hence, $ECG(\Gamma(R)) = ecg(\Gamma(R))$.*

Dominating Sets

Given a connected graph G, we say a subset X of the vertex set of G *dominates* G if each vertex of G is adjacent to some element of X. Trivially, the entire vertex set of G dominates G, unless G is the graph on one vertex (which cannot be dominated). We may use the $\sim$-equivalence classes of G to construct a set that dominates G.

PROPOSITION 49. *Let G be a connected graph on more than one vertex. Construct a subset X of the vertex set of G by selecting one element from each $\sim$-equivalence class of G. Then X dominates G.*

Proof : Let a be a vertex of G. Since $ecg(G)$ is connected, $\overline{a}$ is adjacent to $\overline{y}$ for some vertex y of G. Then a is adjacent to y. There is some element $y_0 \in X$ such that $\overline{y_0} = \overline{y}$. Thus y_0 is adjacent to a. □

Note that if G is a connected graph on more than one vertex, then $ecg(G)$ also has more than one vertex since adjacent vertices cannot be in the same $\sim$-equivalence class. The same is not true for the $\approx$-relation, but we get a similar result.

PROPOSITION 50. *Let G be a connected graph on more than one vertex. If $ECG(G)$ is a graph on more than one vertex, construct a subset X of the vertex set of G by selecting one element from each $\approx$-equivalence class of G. If $ECG(G)$ consists of a single vertex, let $X = \{x, y\}$ for any two distinct vertices x and y of G. Then, in either case, X dominates G.*

Proof : Case 1: $ECG(G)$ has more than one vertex. Since $x \approx y$ implies either x is adjacent to y or $x \sim y$, we can use the proof of the above proposition to show X dominates G.

Case 2: $ECG(G)$ consists of a single vertex. Then G is complete by Proposition 41. Therefore, each vertex of G is adjacent to at least one member of X. □

EXAMPLE 51. *The converse of each of the above two propositions fails: that is, it is possible to find a set X that dominates a graph where X does not intersect every $\sim$-equivalence class or $\approx$-equivalence class of G. Let $R = \mathbb{Z}_{12}$. Note that $\Gamma(R)$ and $ecg(\Gamma(R)) = ECG(\Gamma(R))$ are shown in Figure 33. $\Gamma(R)$ consists of the equivalence classes $\overline{3} = \{3, 9\}$, $\overline{4} = \{4, 8\}$, $\overline{2} = \{2, 10\}$, and $\overline{6} = \{6\}$. The sets $X = \{4, 6\}$ and $Y = \{6, 8\}$ dominate $\Gamma(R)$, but neither set intersects $\overline{2}$.*

For a connected graph G, we say a subset X of the vertex set of G is a *minimal dominating set* of G if X dominates G and no proper subset of X dominates G. The above example gives two distinct minimal dominating sets of $\Gamma(\mathbb{Z}_{12})$. We next show that any minimal dominating set of G contains either 0 or 1 element from each $\sim$-equivalence class of G.

THEOREM 52. *Let G be a connected graph on more than one vertex. Let $Y = \{y_\alpha\}_{\alpha \in A}$ be a minimal dominating set of G. Then, for $\alpha, \beta \in A$, $y_\alpha \not\sim y_\beta$ whenever $\alpha \neq \beta$.*

Proof : Assume $y_\alpha \sim y_\beta$ for some $\alpha, \beta \in A$ with $\alpha \neq \beta$. Consider $X = Y - \{y_\beta\}$. Let a be a vertex of G. If a is adjacent to y_β, then a is adjacent to $y_\alpha \in X$. If a is not adjacent to y_β, then a must be adjacent to some element of X since Y dominates G. Thus we have shown X dominates G, contradicting the minimality of Y. □

COROLLARY 53. *Let G be a connected graph on more than one vertex. Let Y be a minimal dominating set of G. Then $card(Y)$ is less than or equal to the number of $\sim$-equivalence classes of G.*

An Open Question Concerning the Orderings of Vertices

In the previous sections, we defined what it meant for vertices x and y of $\Gamma(R)$ to be "equivalent", denoted $x \sim y$, for any commutative ring R. This idea could be stated in purely ring-theoretic terms as $xy \neq 0$ and $ann(x) = ann(y)$. The following question was raised in considerations of the relation $\sim$ and A. Lauve's work **[13]** mentioned with Figure 36.

Question : Given a commutative ring R and distinct vertices x and y in $\Gamma(R)$ such that $x \sim y$ (i.e., given distinct zero-divisors such that $xy \neq 0$ and $ann(x) = ann(y)$), when is it the case that we can write $nx = my$ for $n, m \in N(R)$ (i.e., n, m are non-zero-divisors of R)?

This section is devoted to giving a partial answer to this question. Before proceeding, we give a definition. An R-module M is *divisible* if for any $u \in M$ and $a \in R$ such that $ann(a) \subseteq ann_M(u)$, then u is divisible by a, i.e., there is some $v \in M$ such that $u = av$. Note that this definition usually involves the "right annihilator of a", but this may be stated as above since we take R to be commutative in this chapter.

THEOREM 54. *Let R be a ring such that R is divisible (viewed as an R-module). Then, if x and y are vertices of $\Gamma(R)$ such that $x \sim y$, we have $xR = yR$.*

Proof : Since R is divisible, $xR = ann(ann(x))$ by [**12**] Proposition 3.17, p. 70. Then $x \sim y$ implies $xR = ann(ann(x)) = ann(ann(y)) = yR$ by Theorem 38. □

COROLLARY 55. *Let R, x, and y be as in the above Theorem with $x \neq y$. Then $x \sim y$ if and only if $x|y$, $y|x$ and $xy \neq 0$.*

Proof : If $x \sim y$, the result follows from the above theorem.

Suppose $x|y$, $y|x$ and $xy \neq 0$. Write $x = ay$ and $y = bx$ for $a, b \in R$. These statements imply, respectively, that $ann(y) \subseteq ann(x)$ and $ann(x) \subseteq ann(y)$. So $ann(x) = ann(y)$. Since $xy \neq 0$ by hypothesis, we have $x \sim y$. □

By an analogous argument, we can prove the following.

COROLLARY 56. *Let R be as in the above Theorem. For vertices x and y of $\Gamma(R)$, $x \approx y$ if and only if $x|y$ and $y|x$.*

COROLLARY 57. *Let R be a von Neumann regular ring. For vertices x and y of $\Gamma(R)$, $x \sim y$ if and only if $xR = yR$.*

Proof : Suppose that $x \sim y$. If R is von Neumann regular, every right R-module is divisible [**12**], Proposition 3.18, p. 71.

Conversely, suppose that $xR = yR$. We can write $x = ys$ for some $s \in R^*$. Therefore, $ann(y) \subseteq ann(x)$. We can also write $y = xt$ for some $t \in R^*$. Thus $ann(x) \subseteq ann(y)$. Hence, $ann(x) - \{x\} = ann(y) - \{y\}$. Thus $x \sim y$ □

The above result, with a different method of proof, appears in [**14**].

EXAMPLE 58. *Let $R = \mathbb{Z} \times \mathbb{Z}_2$. Then R is a ring such that $R_{N(R)}$ is divisible. (Note that R is not von Neumann regular since, for instance, $(2,1) \neq (2,1)(a,b)(2,1)$ for all $(a,b) \in R$.)*

Proof : *Let $S = R_{N(R)}$. Let $a \in R^*$ and let $u \in S^*$ such that $ann(a) \subseteq ann(u)$. Let us write $u = \frac{(x,y)}{(n,1)}$ for $n, x \in \mathbb{Z}$ and $y \in \mathbb{Z}_2$.*
<u>*Case 1*:</u> *$ann(a) = 0$. Then $a \in N(R)$. So we can write $a = (m,1)$ for some $0 \neq m \in \mathbb{Z}$. Then $u = a\frac{u}{a} = (m,1)\frac{(x,y)}{(mn,1)}$.*
<u>*Case 2*:</u> *$ann(a) \neq 0$. If $a = (0,1)$, then $ann(a) = \{(b,0) \mid b \in \mathbb{Z}\}$. Since $ann(u) \supseteq ann(a)$, we must have $x = 0$ and $y = 1$. That is, $u = a$.*
If $a = (m,0)$ for $0 \neq m \in \mathbb{Z}$, then $ann(a) = \{(0,0),(0,1)\}$. Since $ann(u) \supseteq ann(a)$, we must have $y = 0$. Thus $u = a\frac{(x,0)}{(mn,1)}$. □

Corollary 55 gives us a partial answer to our question if R is divisible as an R-module (for example if R is von Neumann regular). We can generalize this as follows.

THEOREM 59. *Let R be a ring and let $N(R)$ be the non-zero-divisors of R. Let $S = R_{N(R)}$. If S is a divisible R-module, then whenever $x \sim y$, $nx \in yR$ and $my \in xR$ for some $n, m \in N(R)$.*

Proof : By [**12**] Proposition 3.17 again, $xS = ann_S(ann(x)) = ann_S(y) = yS$. So $x \in yS$ implies $x = \frac{ay}{n}$ for some $a \in R$ and $n \in N(R)$. Thus $nx \in yR$. Similarly $my \in xR$ for some $m \in N(R)$. □

In the above theorem, we may write $nx = yr$ for some $r \in R$. Since $a \in ann(r)$ implies $a \in ann(x)$, we must have $ann(r) \subsetneq ann(x)$. We cannot necessarily conclude that $ann(r) = 0$, i.e. $r \in N(R)$. Thus, we have not quite satisfied the conditions of the question.

EXAMPLE 60. *Let $R = \mathbb{Z} \times \mathbb{Z}_2$ (refer to Figure 36). Here we have $(2,0) \sim (3,0)$. Let $n = (2,1)$ and $m = (3,1)$. Then $n, m \in N(R)$ and $m(2,0) = n(3,0)$. But also we have $(2,0)(3,0) = (3,0)(2,0)$, where clearly $(2,0),(3,0) \in Z(R)$.*

The Structure of $\Gamma(R \times S)$

One convenient way to produce rings with zero-divisors is to take the Cartesian product of two rings. We introduce the following notation to be used in this section: For a graph G, let

$V(G)$ denote the set of all vertices of G.

$E(G)$ denote the set of all edges of G.

For a ring T, let $n(T,2) = \{x \in T |\ nildeg(x) = 2 \text{ or } x = 0\}$.

PROPOSITION 61. *Let R and S be commutative rings. Define the following subsets of $R \times S$:*

$A = \{(r,0) |\ r \in R^*\}$ $\qquad$ $B = \{(0,s) |\ s \in S^*\}$

$C = \{(r,s') |\ r \in R^*\ s' \in Z(S)^*\}$

$D = \{(r',s) |\ r' \in Z(R)^*\ s \in S^*))\}$

Then $V(\Gamma(R \times S)) = A \cup B \cup C \cup D$ and

$card(V(\Gamma(R \times S))) = (card(R) - 1) + (card(S) - 1) + card(V(\Gamma(R))) \cdot (card(S) - 1) + card(V(\Gamma(S)) \cdot (card(R) - 1) - card(V(\Gamma(R))) \cdot card(V(\Gamma(S)))$.

Proof : For all $a \in A$ and $b \in B$, $ab = (0,0)$. Thus $A, B \subseteq V(\Gamma(R \times S))$. Let $(r,s) \in V(\Gamma(R \times S)) - (A \cup B)$. Then $r \neq 0$ and $s \neq 0$. There is some vertex (r', s') adjacent to (r,s). Thus $rr' = 0$ and $ss' = 0$. If $r' \neq 0$, then $r \in Z(R)^*$ and so $(r,s) \in D$. If $s' \neq 0$, then $s \in Z(S))^*$ and so $(r,s) \in C$. Therefore $V(\Gamma(R \times S)) \subseteq A \cup B \cup C \cup D$.

If $(r', s) \in C$, then $r' \in Z(R)^*$ and so there is some $t \in R^*$ such that $tr' = 0$. Then $(r', s)(t, 0) = (0,0)$. Thus $D \subseteq V(\Gamma(R \times S))$. Similarly, one may show $C \subseteq V(\Gamma(R \times S))$.

The last equation is merely counting the distinct elements in $A \cup B \cup C \cup D$. □

COROLLARY 62. *$\Gamma(R \times S)$ contains a subgraph isomorphic to $K^{|R|-1,|S|-1}$. If R and S are integral domains, $\Gamma(R \times S) \simeq K^{|S|-1,|R|-1}$.*

Proof : The sets A and B as defined above determine a complete bipartite graph. If R and S are integral domains, $V(\Gamma(R \times S)) = A \cup B$. □

COROLLARY 63. *$\Gamma(R \times S)$ contains at least $card(n(R,2))$ disjoint subgraphs isomorphic to $\Gamma(S)$ and $card(n(S,2))$ disjoint subgraphs isomorphic to $\Gamma(R)$.*

Proof : For each $x \in n(R,2)$, take as the vertex set $\{(x,s) | \ s \in V(\Gamma(S))\}$. Since $x^2 = 0$, $(x,s) - (x,s')$ is an edge in $\Gamma(R \times S)$ if and only if $s - s'$ is an edge in $\Gamma(S)$. □

PROPOSITION 64. *Let R and S be rings. Define the following subsets of $(R \times S) \times (R \times S)$:*

$A = \{((r,0),(0,s)) |\ r \in R^*,\ s \in S^*\}$

$B = \{((r,s),(0,s')) |\ r \in R^*,\ s - s' \in E(\Gamma(S))\}$

$C = \{((r,s),(r',0)) |\ r - r' \in E(\Gamma(R)),\ s \in S^*\}$

$D = \{((r,s),(r',s')) |\ r - r' \in E(\Gamma(R)),\ s - s' \in E(\Gamma(S))\}$

$E = \{((x,s),(x,s')) |\ x \in n(R,2),\ s - s' \in E(\Gamma(S))\}$

$F = \{((r,y),(r',y)) |\ r - r' \in E(\Gamma(R)),\ y \in n(S,2)\}$

$G = \{((0,y),(r,y)) |\ r \in R^*,\ 0 \neq y \in n(S,2)\}$

$H = \{((x,0),(x,s)) |\ 0 \neq x \in n(R,2),\ s \in S^*\}$.

Then $E(\Gamma(R \times S)) = A \cup B \cup C \cup D \cup E \cup F \cup G \cup H$ and

$card(E(\Gamma(R \times S))) = (card(R) - 1)(card(S) - 1) +$
$2card(E(\Gamma(S)))(card(R) - 1) + 2card(E(\Gamma(R)))(card(S) - 1) +$
$2card(E(\Gamma(R)))card(E(\Gamma(S))) + card(n(R,2))card(E(\Gamma(S))) +$
$card(n(S,2))card(E(\Gamma(R))) + (card(n(S,2)) - 1)(card(R) - 1) +$
$(card(n(R,2)) - 1)(card(S) - 1)$.

Proof : Clearly, each element of any above subset defines an edge of $\Gamma(R \times S)$, and these subsets are pairwise disjoint. Thus the equation above merely counts the elements in the subsets A through H.

Let $(r,s) - (r',s')$ be an edge between distinct vertices of $\Gamma(R \times S)$. Then $rr' = 0 = ss'$.

<u>Case 1 :</u> $r = 0$ and $s \neq 0$. If $s' = 0$, then $r' \neq 0$ and therefore the edge is in A. For $s' \neq 0$, then the edge is in B or E if $s \neq s'$, and the edge is in G if $s = s'$.

<u>Case 2 :</u> $r \neq 0$ and $s = 0$. Then by an argument similar to that of

Case 1, either the edge is in A, C, F, or H.

<u>*Case 3 :*</u> $r \neq 0$ and $s \neq 0$. If $r' = 0$, then the edge is in B if $s \neq s'$ or in G if $s = s'$. If $s' = 0$, then the edge is in C if $r \neq r'$ or in H if $r = r'$. For the case $r' \neq 0, s' \neq 0$, we have the edge is in E if $r = r'$, or in F if $s = s'$, or in D if both $r \neq r'$ and $s \neq s'$.

As we have now exhausted all cases, we have $e(\Gamma(R \times S)) \subseteq A \cup B \cup C \cup D \cup E \cup F \cup G \cup H$. □

As in Chapter 2, for a graph G we define $\delta(G) = min\{deg(x) | \ x \in V(G)\}$. The next result is a direct consequence of Corollary 62.

COROLLARY 65. *If R and S are integral domains, $\delta(\Gamma(R \times S)) = min\{|S|, |R|\} - 1$.*

PROPOSITION 66. *Let S be an integral domain, and R a ring such that R has at least one nontrivial zero-divisor (that is, R is not an integral domain). Then $\delta(\Gamma(R)) \leq \delta(\Gamma(R \times S)) \leq \delta(\Gamma(R)) + 1$.*

Proof : Let (a, b) be a vertex of $\Gamma(R \times S)$.

<u>*Case 1 :*</u> $b = 0$. Then $a \neq 0$ and $(a, b) - (c, d)$ is an edge implies $c \in Z(R)$ and $d \in S$ is arbitrary. Thus $deg((a, b)) \geq deg_{\Gamma(R)}(a) \cdot |S^*| \geq \delta(\Gamma(R))$.

<u>*Case 2 :*</u> $a = 0$. Then $(a, b) - (c, 0)$ is an edge of $\Gamma(R \times S)$ for any $c \in R^*$. Thus $deg((a, b)) \geq |R^*| \geq \deg_{\Gamma(R)}(a) \geq \delta(\Gamma(R))$.

<u>*Case 3 :*</u> $a \neq 0, b \neq 0$. If $(a, b) - (c, d)$ is an edge of $\Gamma(R \times S)$, then $d = 0$ and $ac = 0$. Thus, $deg((a, b)) = deg_{\Gamma(R)}(a)$ if $a^2 \neq 0$, and $deg((a, b)) = deg_{\Gamma(R)}(a) + 1$ if $a^2 = 0$ (since here we get the additional

vertex $(a, 0)$). Note that in either circumstance, $deg((a, b)) \geq \delta(\Gamma(R))$, and we are always able to find some vertex of $\Gamma(R \times S)$ as in this case.

Hence, we have shown in every possible case that $deg((a, b)) \geq \delta(\Gamma(R))$.

Let $A = \{x \in Z(R)^* | \ deg_{\Gamma(R)}(x) = \delta(\Gamma(R))\}$. If $x^2 \neq 0$ for some $x \in A$, then $\delta(\Gamma(R \times S)) = deg_{\Gamma(R)}(x) = \delta(\Gamma(R))$. If $x^2 = 0$ for all $x \in A$, then $\delta(\Gamma(R \times S)) = deg_{\Gamma(R)}(x) + 1 = \delta(\Gamma(R)) + 1$. □

PROPOSITION 67. *Let R and S be rings, each of which contains a nontrivial zero-divisor (that is, neither R nor S is an integral domain). Then*

$min\{\delta(\Gamma(R)), \delta(\Gamma(S))\} \leq \delta(\Gamma(R \times S)) \leq min\{\delta(\Gamma(R)), \delta(\Gamma(S))\} + 1.$

Proof : Recall, we let $N(R)$ be the set of non-zero-divisors of R.

Let (a, b) be a vertex of $\Gamma(R \times S)$.

<u>Case 1 :</u> $a = 0$. Then $(a, b) - (r, 0)$ is an edge for each $0 \neq r \in R$. Thus $deg((a, b)) \geq |R^*| \geq deg_{\Gamma(R)}(a) \geq \delta(\Gamma(R))$.

<u>Case 2 :</u> $b = 0$. By an argument analogous to the above case, $deg((a, b)) \geq \delta(\Gamma(S))$.

<u>Case 3 :</u> $a \in Z(R)^*, b \in N(S)$. Then $(a, b) - (c, d)$ is an edge of $\Gamma(R \times S)$ if and only if $d = 0$ and either c is adjacent to a in $\Gamma(R)$ or $a^2 = 0$ and $c = a$. Thus $deg((a, b)) = deg_{\Gamma(R)}(a) \geq \delta(\Gamma(R))$ if $a^2 \neq 0$ or $deg((a, b)) = deg_{\Gamma(R)}(a) + 1 \geq \delta(\Gamma(R)) + 1$ if $a^2 = 0$.

<u>Case 4 :</u> $a \in N(R), b \in Z(R)^*$. By an argument analogous to the above case we get $deg((a, b)) = deg_{\Gamma(S)}(b) \geq \delta(\Gamma(S))$ if $b^2 \neq 0$ and $deg((a, b)) = deg_{\Gamma(S)}(b) + 1 \geq \delta(\Gamma(S)) + 1$ if $b^2 = 0$.

<u>Case 5 :</u> $a \in Z(R)^*, b \in Z(S)^*$. Then $(a, b) - (c, d)$ is an edge in $\Gamma(R \times$

S) whenever $ac = 0$ and $bd = 0$; in particular, whenever c is adjacent to a and d is adjacent to b. Thus $deg((a,b)) \geq deg_{\Gamma(R)}(a)deg_{\Gamma(S)}(b) \geq min\{\delta(\Gamma(R)), \delta(\Gamma(S))\}$.

Hence, we have shown that in every possible case $deg((a,b)) \geq min\{\delta(\Gamma(R)), \delta(\Gamma(S))\}$.

Let $A = \{a \in R^* |\, deg_{\Gamma(R)}(a) = \delta(\Gamma(R))\}$ and $B = \{b \in S^* |\, deg_{\Gamma(S)}(b) = \delta(\Gamma(S))\}$. If $\delta(\Gamma(R)) < \delta(\Gamma(S))$, then either $\delta(\Gamma(R \times S)) = deg((a,1)) = \delta(\Gamma(R))$ if there is some $a \in A$ such that $a^2 \neq 0$, or $\delta(\Gamma(R \times S)) = deg((a,1)) = \delta(\Gamma(R)) + 1$ if $a^2 = 0$ for all $a \in A$. Similarly, if $\delta(\Gamma(S)) < \delta(\Gamma(R))$, we get $\delta(\Gamma(R \times S)) = \delta(\Gamma(S))$ or $\delta(\Gamma(S)) + 1$. If $\delta(\Gamma(S)) = \delta(\Gamma(R))$, a similar argument shows either $\delta(\Gamma(R \times S)) = \delta(\Gamma(S))$ or $\delta(\Gamma(S)) + 1$. □

More on the Degree of a Vertex

Recall that an edge E of a connected graph G is called a bridge if $G - E$ is disconnected. In the case of $\Gamma(R)$, one can connect the concept of a bridge to the previous relations on the vertices of the graph.

PROPOSITION 68. *Let G be a connected graph. If the edge $a - b$ is a bridge of G, then $a \perp b$.*

Proof : Clearly, a is adjacent to b since $a - b$ is an edge of G. If G contains a vertex c adjacent to both a and b, then $a - c$ and $c - b$ are edges of the graph $G - \{a - b\}$. Thus any path containing the edge $a - b$ can be replaced with path containing $a - c$ and $c - b$. Thus, $G - \{a - b\}$ is connected. However, this contradicts the fact that $a - b$ is a bridge. □

Figure 40 shows that the converse of this Proposition may fail.

A result analogous to the following appears on page 444 of [2].

PROPOSITION 69. *Let R be a ring. Then $\delta(\Gamma(R))$ is either $min\{|ann(x)| : x \in Z(R)^*\} - 1$, or $min\{|ann(x)| : x \in Z(R)^*\} - 2$.*

Proof : For any vertex x of $\Gamma(R)$, the vertices adjacent to x are precisely those $a \in ann(x)$ with $a \neq 0$ and $a \neq x$. Thus, either $deg(x) = |ann(x)| - 1$ if $x^2 \neq 0$, or $deg(x) = |ann(x)| - 2$ if $x^2 = 0$. Let $M = min\{|ann(x)| : x \in R\}$ and let $A = \{x \in R^* | \ deg(x) = M\}$. If there exists some $a \in A$ such that $a^2 = 0$, then $\delta(\Gamma(R)) = M - 2$. Otherwise, $\delta(\Gamma(R)) = M - 1$. □

COROLLARY 70. *For any vertex x of $\Gamma(R)$,*

$$deg(x) = \begin{cases} |ann(x)| - 1 & \text{if } x^2 \neq 0, \\ |ann(x)| - 2 & \text{if } x^2 = 0. \end{cases}$$

If R is reduced, then $\delta(\Gamma(R)) = min\{|ann(x)| : x \in R\} - 1$.

For the next few results, we use the following lemma.

LEMMA 71. *Let R be a ring with $char(R) = m \neq 0$ and let $0 \neq x \in R$. If $n \geq 2$ is the smallest integer such that $nx = 0$, then n divides m.*

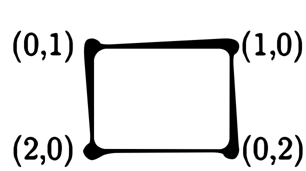

FIGURE 40. Given below is $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$, where $(1,0) \perp (0,1)$ but $(1,0) - (0,1)$ is not a bridge.

Proof : The order of x in the group $< R, + >$ is n. Then $char(R) = m$ implies $mx = 0$. Thus $n|m$. □

THEOREM 72. *Let R be a ring such that $\delta(\Gamma(R)) = 1$. Then either $\Gamma(R)$ is a graph on two vertices, or there exists some $0 \neq a \in R$ such that $2a = 0$.*

Proof : If $\delta(\Gamma(R)) = 1$, there is some $x \in R^*$ with $deg(x) = 1$. So there is a unique $y \in R^*$ such that $x \neq y$ and $xy = 0$. Now $x(y + y) = 0$. Thus either $2y = x$ or $2y = 0$. If $2y = 0$, we are done. If $2y = x$, we show $\Gamma(R)$ is a graph on two vertices. Now, $2y = x$ implies $x(x + y) = x^2 + xy = x(2y) + 0 = 0$. That is, $y = -x$. So, if $\Gamma(R)$ has a vertex a adjacent to y, a is also adjacent to x. But, since $deg(x) = 1$, we must have $a = x$. Hence, $\Gamma(R)$ has only the vertices x and $-x$. □

EXAMPLE 73. *$\Gamma(\mathbb{Z}_{12})$, given in Figure 33, is graph on more than two vertices. Note that $\delta(\Gamma(\mathbb{Z}_{12})) = 1$ and $6 + 12\mathbb{Z} \in \mathbb{Z}_{12}$ with $2(6 + 12\mathbb{Z}) = 0$.*

COROLLARY 74. *Let R be a ring such that $\Gamma(R)$ is a graph on three or more vertices. If $deg(x) = 1$ for some vertex x of $\Gamma(R)$, then $char(R)$ is even (possibly zero).*

EXAMPLE 75. *$\Gamma(\mathbb{Z} \times \mathbb{Z}_2)$, given in Figure 36, is the graph of a ring of characteristic zero with $\delta(\Gamma(\mathbb{Z} \times \mathbb{Z}_2)) = 1$.*

COROLLARY 76. *Let R be a ring such that $char(R)$ is odd and $\Gamma(R)$ is a graph on three or more vertices. Then $\delta(\Gamma(R)) > 1$.*

COROLLARY 77. *Let R be a ring with $char(R) = m \neq 0$. If neither 2 nor 3 divides m, then $\Gamma(R)$ is not the connected graph on two vertices.*

Proof : Assume $\Gamma(R)$ consists only of the vertices x and y. Since 2 does not divide m, $x \neq -x$ and $y \neq -y$. Thus, we must have $x = -y$. Now, $2y \neq 0$ and $2y \in ann(x) \subseteq \{0, x, y\}$. Since $2y \neq y$, we must have $2y = x = -y$. Thus $3y = 0$. But this contradicts the fact that 3 does not divide m. □

A list of all rings R (up to isomorphism) such that $\Gamma(R)$ is the connected graph on two vertices is given in Example 2.1 (a) of [**3**].

We can generalize the above theorem as follows.

THEOREM 78. *Let R be a ring. If $\delta(\Gamma(R)) = n$ for some positive integer n, then there is some $0 \neq a \in R$ such that $ma = 0$ for some integer $2 \leq m \leq n+2$.*

Proof : Since $\delta(\Gamma(R)) = n$, there is a vertex x of $\Gamma(R)$ with $deg(x) = n$. Thus $ann(x) - \{0, x\} = \{x_1, x_2, \ldots, x_n\}$. For any $m \in \mathbb{Z}$, $(mx_1)x = 0$. So $mx_1 \in \{x_1, \ldots, x_n, 0, x\}$ for all $m = 1, 2, \ldots, n+1$. If $mx_1 = 0$ for some $m = 1, 2, \ldots, n+1$, we are done. If this is not the case, then each mx_1 is distinct. Since $\{mx_1 \mid m = 1, \ldots, n+2\} \subseteq \{x_1, \ldots, x_n, 0, x\}$, we must have $(n+2)x_1 = 0$ (if $(n+2)x_1 = mx_1$ for $1 \leq m \leq n+1$, then $(n-m+2)x_1 = 0$). □

COROLLARY 79. *Let R be a ring with $char(R) = m \neq 0$, such that $\Gamma(R)$ is a graph on three or more vertices. If k is a positive integer such that $gcd(n, m) = 1$ for all $2 \leq n \leq k$, then $\delta(\Gamma(R)) \geq k-1$.*

Proof : The cases for $k = 1, 2$ are trivial. If $\delta(\Gamma(R)) \leq k - 2$, then R has an element of additive order $2 \leq n \leq k$. Thus n divides $char(R) = m$, a contradiction. □

COROLLARY 80. *Let R be a ring with $char(R) = m \neq 0$, such that $\Gamma(R)$ is a graph on three or more vertices. Let p be the smallest prime dividing m. Then $\delta(\Gamma(R)) \geq p - 2$.*

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Appendix

Let R and S be commutative rings with identity. Figures 41 through 54 give the equivalence class graph of $\Gamma(R \times S)$, $ecg(\Gamma(R \times S))$, based on the properties of $\Gamma(R)$ and $\Gamma(S)$.

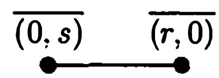


FIGURE 41. R and S are both integral domains. For $r \in R^*$ and $s \in S^*$:

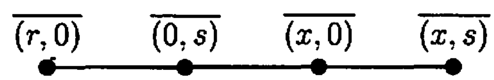


FIGURE 42. S is an integral domain and $\Gamma(R) = \{x\}$. Then $x^2 = 0$ in R. For $r \in R - \{0, x\}$ and $s \in S^*$:

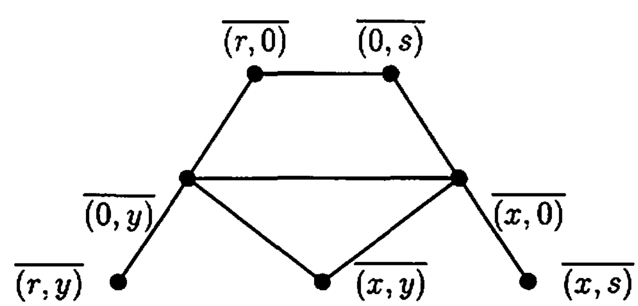


FIGURE 43. $\Gamma(R) = \{x\}$ and $\Gamma(S) = \{y\}$. For $r \in R - \{0, x\}$ and $s \in S - \{0, y\}$:

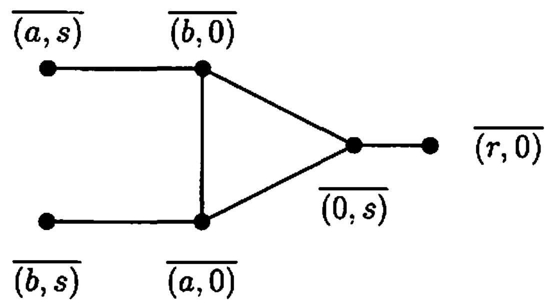


FIGURE 44. $\Gamma(R)$ has vertices a and b with $a^2 \neq 0$ and $b^2 \neq 0$, and S is an integral domain. For $r \in R - \{0, a, b\}$ and $s \in S^*$:

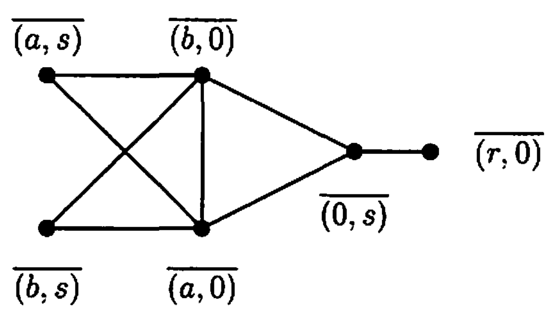


FIGURE 45. $\Gamma(R)$ has vertices a and b with $a^2 = b^2 = 0$, and S is an integral domain. For $r \in R - \{0, a, b\}$ and $s \in S^*$:

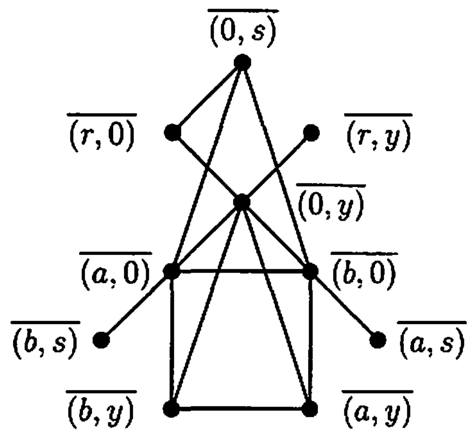


FIGURE 46. $\Gamma(R)$ has vertices a and b with $a^2 \neq 0$ and $b^2 \neq 0$, and $\Gamma(S) = \{y\}$. Then $y^2 = 0$. For $r \in R - \{0, a, b\}$ and $s \in S - \{0, y\}$:

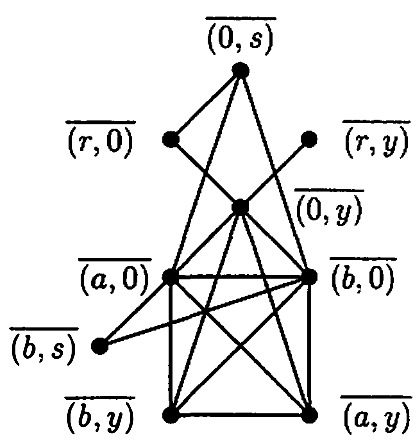
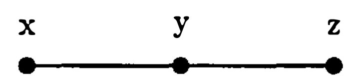
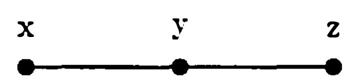


FIGURE 47. $\Gamma(R)$ has vertices a and b with $a^2 = b^2 = 0$, and $\Gamma(S) = \{y\}$. Then $y^2 = 0$. In this case $(a, s) \sim (b, s)$ for all $s \in S^*$. For $r \in R - \{0, a, b\}$ and $s \in S - \{0, y\}$:

Type 1: $y^2 \neq 0,\ x^2 \neq 0,\ z^2 \neq 0$



Type 2: $y^2 = 0,\ x^2 \neq 0,\ z^2 \neq 0$



Type 3: $y^2 = x^2 = z^2 = 0$

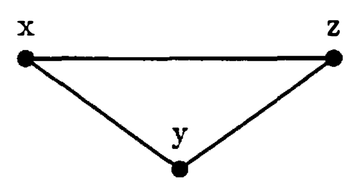


FIGURE 48. Recall, for a commutative ring R there are three possibilities if $\Gamma(R)$ is a graph on three vertices.

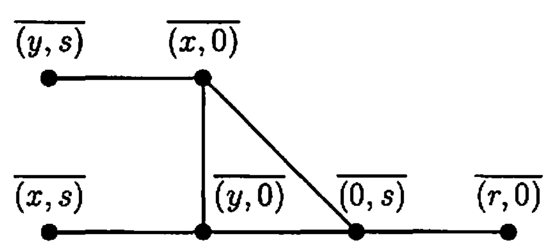


FIGURE 49. $\Gamma(R)$ has vertices x, y, and z and is of Type 1, and S is an integral domain. We have $(x,s) \sim (z,s)$ for all $s \in S$. For $r \in R - \{0,x,y,z\}$ and $s \in S^*$:

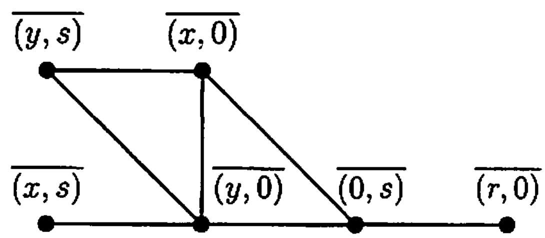


FIGURE 50. $\Gamma(R)$ has vertices x, y, and z and is of Type 2, and S is an integral domain. We have $(x, s) \sim (z, s)$ for all $s \in S$. For $r \in R - \{0, x, y, z\}$ and $s \in S^*$:

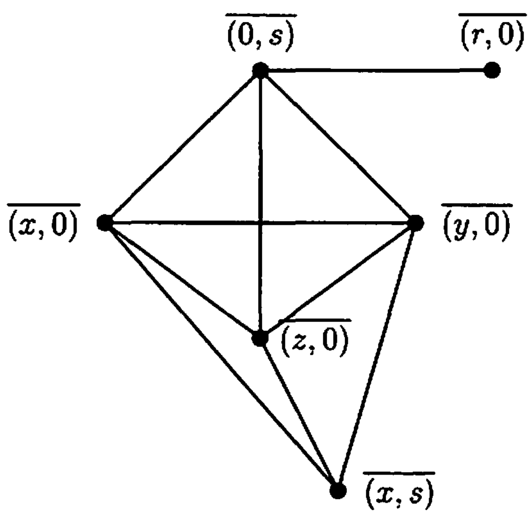


FIGURE 51. $\Gamma(R)$ has vertices x, y, and z and is of Type 3, and S is an integral domain. We have $(x, s) \sim (z, s) \sim (y, s)$ for all $s \in S^*$. For $r \in R - \{0, x, y, z\}$ and $s \in S^*$:

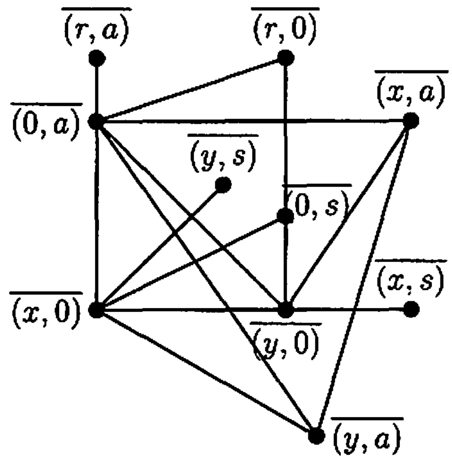


FIGURE 52. $\Gamma(R)$ has vertices x, y, and z and is of Type 1, and $\Gamma(S) = \{a\}$. Then $a^2 = 0$. We have $(x,s) \sim (z,s)$ for all $s \in S$. For $r \in R - \{0,x,y,z\}$ and $s \in S - \{0,a\}$:

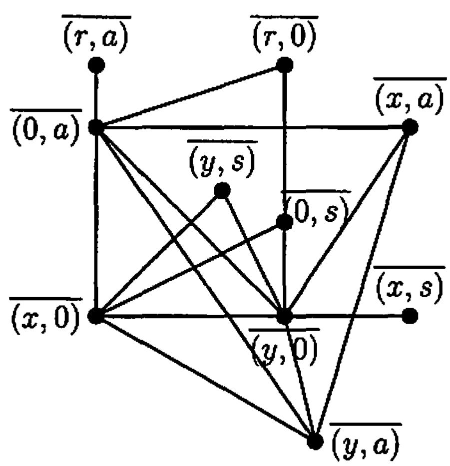


FIGURE 53. $\Gamma(R)$ has vertices x, y, and z and is of Type 2, and $\Gamma(S) = \{a\}$. Then $a^2 = 0$. We have $(x,s) \sim (z,s)$ for all $s \in S$. For $r \in R - \{0,x,y,z\}$ and $s \in S - \{0,a\}$:

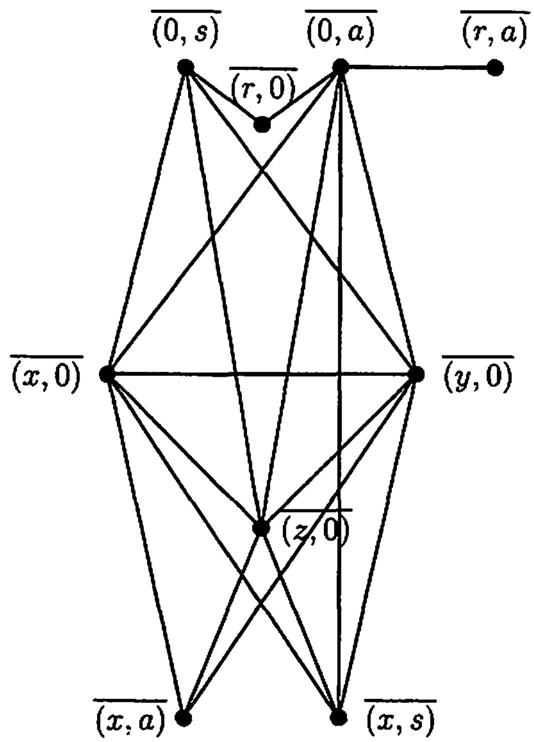


FIGURE 54. $\Gamma(R)$ has vertices x, y, and z and is of Type 3, and $\Gamma(S) = \{a\}$. Then $a^2 = 0$. We have $(x, s) \sim (z, s) \sim (y, s)$ for all $s \in S^*$. For $r \in R - \{0, x, y, z\}$ and $s \in S - \{0, a\}$:

Vita

Shane Patrick Redmond was born in Mansfield, Ohio on June 11, 1972. He graduated from Mansfield Senior High School in 1990. He attended Ohio University in Athens, Ohio. In 1993, he served an internship at Argonne National Labs in Argonne, Illinois. He received the Bachelor of Science degree in mathematics, with minors in computer science and history, in 1994 from Ohio University. He returned to Ohio University to earn the Master of Science degree in mathematics in 1996. In August 1996, he entered The University of Tennessee, Knoxville to pursue the Doctorate of Philosophy in Mathematics. The doctoral degree was received in May 2001 under David F. Anderson. He has accepted a tenure-track faculty position with Southeastern Louisiana University in Hammond, Louisiana beginning in August 2001.